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THE GALOIS STRUCTURE OF AMBIGUOUS IDEALS IN CYCLIC EXTENSIONS OF DEGREE 8

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Abstract. In cyclic, degree 8 extensions of algebraic number fields N/K, ambiguous ideals in N are canonical $\mathbb{Z}[C_8]$-modules. Their $\mathbb{Z}[C_8]$-structure is determined here. It is described in terms of indecomposable modules and determined by ramification invariants. Although infinitely many indecomposable $\mathbb{Z}[C_8]$-modules are available (classification by Yakovlev), only 23 appear.

1. Introduction

We are concerned with the interrelationship between two basic objects in algebraic number theory: the *ring of integers* and the *Galois group*. In particular, we seek to understand the *effect* of the Galois group upon the ring of integers. At the same time, we are also interested in the Galois action upon other fractional ideals. So that the action may be similar, we restrict ourselves to *ambiguous ideals* – those that are mapped to themselves by the Galois group. The setting for our investigation is the family of C_8-extensions. This choice is guided by by a result of E. Noether as well as results in Integral Representation Theory.

Noether's Normal Integral Basis Theorem. A finite Galois extension of number fields N/K is said to be *at most tamely ramified* (TAME) if the factorization of each prime ideal $\mathfrak{P}_K$ (of $\mathfrak{O}_K$) in $\mathfrak{O}_N$ results in exponents (degrees of ramification) that are relatively prime to the ideal $\mathfrak{P}_K$. A *normal integral basis* (NIB) is said to exist if there is an element $\alpha \in \mathfrak{O}_N$ (in the ring of integers of N) whose conjugates, $\{\sigma\alpha : \sigma \in \mathrm{Gal}(N/K)\}$, provide a basis for $\mathfrak{O}_N$ over $\mathfrak{O}_K$ (the integers in K).

Noether proved NIB $\Rightarrow$ TAME; moreover, for local number fields NIB $\Leftrightarrow$ TAME, tying the Galois module structure of the ring of integers to the arithmetic of the extension [Noe32]. This is a *nice* effect – NIB means that the integers are isomorphic to the group ring, $\mathfrak{O}_K[\mathrm{Gal}(N/K)]$. It is similar to the effect of the Galois group on the field itself (*i.e.* Normal Basis Theorem). The impact of her result is two-fold: (1) We are encouraged to localize. (2) We are directed away from tamely ramified extensions – toward wildly ramified extensions and p-groups (See [Miy87]).

Integral Representation Theory (Restricted to p-groups G).

Classification of Modules. The number of indecomposable modules over a group ring $\mathbb{Z}[G]$ is, in general, infinite. Only $\mathbb{Z}[C_p]$ and $\mathbb{Z}[C_{p^2}]$ are of *finite type*. Still, among those of *infinite type*, there are two whose classifications are somehow manageable. These are the ones of so–called *tame type* [Die85]: $\mathbb{Z}[C_2 \times C_2]$ (classification by L. A. Nazarova [Naz61]) and $\mathbb{Z}[C_8]$ (classification by A. V. Yakovlev [Jak75]).

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Unique Decomposition. The Krull–Schmidt–Azumaya Theorem does not, in general, hold: although a module over a group ring will decompose into indecomposable modules, this decomposition may not be unique. Fortunately, it does hold for a few group rings, including $\mathbb{Z}[C_2 \times C_2]$ and $\mathbb{Z}[C_8]$ [HKO98].
Topic. Let $G = \mathrm{Gal}(N/K)$. We are led to ask the following natural question: *What is the $\mathbb{Z}[G]$-module structure of ambiguous ideals when*

- *the number theory is 'bad' (wild ramification), while*
- *the representation theory is 'good' (tame type, K–S–A)?*

In other words: *What is the $\mathbb{Z}[G]$-module structure of ambiguous ideals in wildly ramified $C_2 \times C_2$ and C_8 number field extensions?* Previous work solved this for $C_2 \times C_2$-extensions [Eld98], [BE02]. So our focus here is on C_8-extensions. (Note: This question has already been addressed for those group rings with 'very good' representation theory, those of *finite type*. See [RCVSM90] and [Eld95].)

As with $C_2 \times C_2$-extensions, the $\mathbb{Z}[G]$-module structure of ambiguous ideals in C_8-extensions is completely determined by the structure at its 2-adic completion – our global question reduces to a collection of local ones. We leave it to the reader to fill in the details. (One may follow [Eld98, §2] using [Wie84].)

1.1. **Local Question, Answer.** Let K_0 be a finite extension of the 2-adic numbers $\mathbb{Q}_2$ and let K_n be a wildly ramified, cyclic, degree 2^n extension of K_0 with $G = \mathrm{Gal}(K_n/K_0)$. The maximal ideal $\mathfrak{P}_n$ in K_n is unique (therefore ambiguous). So every fractional ideal $\mathfrak{P}_n^i$ is ambiguous. We ask: *What is the $\mathbb{Z}_2[G]$-module structure of $\mathfrak{P}_n^i$ for $n = 1, 2, 3$?* ($\mathbb{Z}_2$ denotes the 2-adic integers.) The answered is given by the following theorem and the description of the modules $\mathfrak{M}_s(i, b_1, \cdots, b_s)$.

Let T denote the maximal unramified extension of $\mathbb{Q}_2$ in K_0. Following [Ser79, Ch IV], let $G = G_{-1} \supseteq G_0 \supseteq G_1 \supseteq \cdots$ denote the ramification filtration. Use subscripts to denote field of reference, so $\mathfrak{O}_k$ denotes the ring of integers of k.

Theorem 1.1. *Let K_n/K_0 be a cyclic extension of degree 2^n and let $k \subseteq K_0$ be an unramified extension of $\mathbb{Q}_2$. Suppose that $|G_1| \leq 8$ (i.e. $s = 1, 2$ or 3) and let $b_1, \cdots, b_s$ be the break numbers in the ramification filtration of G_1. If $\mathfrak{M}_s(i, b_1, \cdots, b_s)$ is the $\mathbb{Z}_2[G_1]$-module defined below, then*

$$\mathfrak{P}_n^i \cong \mathfrak{O}_k[G] \otimes_{\mathbb{Z}_2[G_1]} \mathfrak{M}_s(i, b_1, \cdots, b_s)^{[T:k]} \text{ as left } \mathfrak{O}_k[G]\text{-modules.}$$

1.1.1. $\mathfrak{M}_s(i, b_1, \cdots, b_s)$. Indecomposable modules are listed in Appendix A and e_0 denotes the absolute ramification index of K_0. Following [RCVSM90] and [Eld95],

$$\mathfrak{M}_1(i, b_1) = (\mathcal{R}_0 \oplus \mathcal{R}_1)^{\lceil (i+b_1)/2 \rceil - \lceil i/2 \rceil} \oplus \mathbb{Z}_2[G_1]^{e_0 - (\lceil (i+b_1)/2 \rceil - \lceil i/2 \rceil)} \tag{1.1}$$

$$\mathfrak{M}_2(i, b_1, b_2) = \mathcal{I}^a \oplus \begin{cases} \mathcal{H}^{b_A} \oplus \mathcal{G}^{c_A} \oplus \mathcal{L}^{d_A} & \text{if } b_2 + 2b_1 > 4e_0, \\ \mathcal{H}^{b_B} \oplus \mathcal{M}^{c_B} \oplus \mathcal{L}^{d_B} & \text{if } b_2 + 2b_1 < 4e_0. \end{cases} \tag{1.2}$$

where $a = \lceil (i + b_2)/4 \rceil - \lceil (i + 2b_1)/4 \rceil$, $b_A = e_0 + \lceil i/4 \rceil - \lceil (i + b_2)/4 \rceil$, $b_B = \lceil (i + b_2 + 2b_1)/4 \rceil - \lceil (i + b_2)/4 \rceil$, $c_A = \lceil (i + b_2 + 2b_1)/4 \rceil - e_0 - \lceil i/4 \rceil$, $c_B = e_0 + \lceil i/4 \rceil - \lceil (i + b_2 + 2b_1)/4 \rceil$, $d_A = e_0 + \lceil (i + 2b_1)/4 \rceil - \lceil (i + b_2 + 2b_1)/4 \rceil$, $d_B = \lceil (i + 2b_1)/4 \rceil - \lceil i/4 \rceil$.

The description of $\mathfrak{M}_3(i, b_1, b_2, b_3)$ is given by Tables 1 and 2. Note the eight columns in each table. There are eight cases. Each module that appears in $\mathfrak{M}_3(i, b_1, b_2, b_3)$, except for $\mathcal{R}_3$, is listed in the appropriate column of Table 1. The

multiplicity of the module is appears in the corresponding spot in Table 2. The multiplicity of $\mathcal{R}_3$ follows the tables.

Table 1.

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
$\mathcal{H}$	$\mathcal{H}$	$\mathcal{H}$	$\mathcal{H}$	$\mathcal{I}_1$	$\mathcal{I}_1$	$\mathcal{I}_1$	$\mathcal{I}_1$
		$\mathcal{H}_1\mathcal{L}$	$\mathcal{H}_1\mathcal{L}$	$\mathcal{H}_1\mathcal{L}$	$\mathcal{H}_1\mathcal{L}$		
			$\mathcal{H}_1\mathcal{G}$		$\mathcal{H}_1\mathcal{G}$		
$\mathcal{H}_2$	$\mathcal{H}_2$	$\mathcal{H}_1$	$\mathcal{H}_1$	$\mathcal{H}_1$	$\mathcal{H}_1$	$\mathcal{H}_1$	$\mathcal{H}_1$
$\mathcal{M}$	$\mathcal{H}_{1,2}$	$\mathcal{H}_{1,2}$	$\mathcal{H}_{1,2}$	$\mathcal{G}$	$\mathcal{G}$	$\mathcal{G}$	$\mathcal{G}$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{G}_4$	$\mathcal{G}_4$	$\mathcal{G}_4$	$\mathcal{G}_4$	$\mathcal{G}_4$	$\mathcal{G}_4$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{G}_3$	$\mathcal{G}_3$	$\mathcal{G}_3$	$\mathcal{G}_3$
$\mathcal{L}_3$	$\mathcal{L}_3$	$\mathcal{L}_3$	$\mathcal{L}_3$	$\mathcal{L}_3$	$\mathcal{L}_3$	$\mathcal{G}_2$	$\mathcal{G}_2$
$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{L}_2$	$\mathcal{I}$	$\mathcal{L}_2$	$\mathcal{L}_2$	$\mathcal{G}_1$
$\mathcal{I}_2$	$\mathcal{I}_2$	$\mathcal{I}_2$	$\mathcal{I}_2$	$\mathcal{I}_2$	$\mathcal{I}_2$	$\mathcal{L}_1$	$\mathcal{L}_1$

Table 2.

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
$\bar{d}-b$	$\bar{d}-b$	$\bar{a}-b-e_0$	$\bar{a}-b-e_0$	$b+e_0-\bar{a}$	$b+e_0-\bar{a}$	$b-c$	$b-c$
		$w+e_0-\bar{a}$	$\bar{y}+m-\bar{a}$	$w-b$	$\bar{y}+m-e_0-b$		
			$\bar{d}-\bar{y}+e_0$		$\bar{c}-\bar{y}$		
$d+e_0-\bar{d}$	$\bar{a}-\bar{d}-e_0$	$\bar{d}-w$	$\bar{d}-d-m$	$a-w$	$a+e_0-\bar{c}-m$	$a-b$	$a-b$
$\bar{a}-d-2e_0$	$d+2e_0-\bar{a}$	$a-\bar{d}$	$a-\bar{d}$	$\bar{d}-a$	$\bar{d}-a$	$\bar{d}-a$	$\bar{d}-a$
$a+e_0-\bar{a}$	$a-d-e_0$	$d+e_0-a$	$\bar{w}-m-a$	$\bar{c}-\bar{d}$	$\bar{y}-\bar{d}$	$\bar{c}-\bar{d}$	$\bar{c}-\bar{d}$
$\bar{c}-a$	$\bar{c}-a$	$\bar{c}-d-e_0$	$\bar{c}-d-e_0$	$d+e_0-\bar{c}$	$d+e_0-\bar{c}$	$\bar{b}-\bar{c}$	$\bar{b}-\bar{c}$
$c+e_0-\bar{c}$	$c+e_0-\bar{c}$	$\bar{z}+b_1-\bar{c}$	$\bar{z}+b_1-\bar{c}$	$y-d$	$y+e_0-d$	$d+e_0-\bar{b}$	$\bar{a}-\bar{b}$
$\bar{b}-c-e_0$	$\bar{b}-c-e_0$	$\bar{b}-c-e_0$	$c+e_0-\bar{b}$	$\bar{b}-c-e_0$	$c+e_0-\bar{b}$	$d+e_0-\bar{b}$	$d+e_0-\bar{a}$
$b-\bar{b}+e_0$	$b-\bar{b}+e_0$	$b-\bar{b}+e_0$	$b-c$	$\bar{a}-\bar{b}$	$\bar{a}-c-e_0$	$c+e_0-\bar{a}$	$c-d$

Cases.

A. $4e_0-4b_1/3<b_2$ (including *Stable Ramification*, $b_1\geq e_0$).

B. $4e_0-2b_1<b_2<4e_0-4b_1/3$

C. $4e_0 - 4b_1 < b_2 < 4e_0 - 2b_1$ and $b_2 > (4e_0 + 4b_1)/3$
D. $4e_0 - 4b_1 < b_2 < 4e_0 - 2b_1$ and $b_2 < (4e_0 + 4b_1)/3$
E. $b_2 < 4e_0 - 4b_1$ and $b_3 > 8e_0 + 4b_1 - 2b_2$
F. $b_2 < 4e_0 - 4b_1$ and $8e_0 + 4b_1 - 2b_2 < b_3 < 8e_0 + 4b_1 - 2b_2$
G. $8e_0 - 4b_1 - 2b_2 < b_3 < 8e_0 - 2b_2$
H. $b_3 < 8e_0 - 4b_1 - 2b_2$

A graphic representation of these cases appears in §3.2.

Constants used in Table 2. $a := \lceil (i - 2b_2)/8 \rceil$, $\bar{a} := \lceil (i + b_3 - 2b_2)/8 \rceil$, $b := \lceil (i - 2b_2 - 4b_1)/8 \rceil$, $\bar{b} := \lceil (i + b_3 - 2b_2 - 4b_1)/8 \rceil$, $c := \lceil (i - 4b_2)/8 \rceil$, $\bar{c} := \lceil (i + b_3 - 4b_2)/8 \rceil$, $d := \lceil (i - 4b_2 - 4b_1)/8 \rceil$, $\bar{d} := \lceil (i + b_3 - 4b_2 - 4b_1)/8 \rceil$, $w := \lceil (i - 2b_2 - 2b_1)/8 \rceil$, $\bar{w} := \lceil (i + b_3 - 2b_2 - 2b_1)/8 \rceil$, $y := \lceil (i - 4b_2 - 2b_1)/8 \rceil$, $\bar{y} := \lceil (i + b_3 - 4b_2 - 2b_1)/8 \rceil$, $\bar{z} := \lceil (i + b_3 - 4b_2 - 6b_1)/8 \rceil$, $m := (b_2 - b_1)/2$.

The multiplicity of $\mathcal{R}_3$. In Cases A and B it is $((\bar{a}+\bar{b}+\bar{c}+\bar{d}) - (a+b+c+d) - 3e_0)f$. In Case C, it is $((\bar{a}+\bar{b}+\bar{c}+\bar{d}) - (a+b+d) - 2e_0 - (\bar{z}+b_1))f$. While in Case D it is $((\bar{a}+\bar{b}+\bar{c}+\bar{d}) - (a+b) - e_0 + m - (\bar{w}+\bar{z}+b_1))f$. In Case E, it is $(\bar{b}+\bar{d}-a-y-e_0)f$. In Case F it is $((\bar{b}+\bar{d}+\bar{c}) - (a+y+\bar{y}) - e_0)f$. Finally, in Cases G and H, the number of $\mathcal{R}_3$ that appear is $(\bar{d} - a)f$.

1.2. **Discussion.**

Cyclic p-Extensions. The Galois module structure of the ring of integers in fully and wildly ramified, cyclic, local extensions of degree p^n was studied in [EM94] and more recently in [Eld02]. Both of these papers required a lower bound on the first ramification number b_1. In particular, [Eld02] restricted b_1 to about half of its possible values, under so-called *strong ramification.* In this paper, by focusing on $p = 2$ we are able to remove this restriction. Our work sheds light (1) on *strong ramification* and (2) on the structures that are possible outside of it.

(1) *Strong ramification* for $p = 2$ means $b_1 > e_0$, a small part of Case A. The structure under *strong ramification* given by [Eld02, Thm 5.3], when restricted to $p = 2$, remains valid throughout Case A. *What then should Case A be, for odd p?*

(2) Suppose that 'nice' refers to the structure under *strong ramification*, indeed under Case A. Does the structure remain relatively 'nice' beyond Case A? This depends upon a precise definition. Let an indecomposable module be *nice* if it is made up of distinct irreducible modules. Note only *nice* modules appear in Case A. But then, as we leave Case A, the structure turns *nasty* immediately. At least one of $\mathcal{H}_{1,2}$, $\mathcal{H}_1\mathcal{L}$ and $\mathcal{H}_1\mathcal{G}$ appears in every Case B through F.

Induced Structure. The subfield of K_n fixed by the first ramification group G_1 is tame over the base field K_0. Miyata generalized Noether's Theorem proving that each ideal is *relatively projective* over G_1 [Miy87]. In other words, the ideals are direct summands of modules that have been induced from G_1 to G [CR90, §10]. We find, in our situation, that ideals are *relatively free* over G_1. See [Miy95, Thm 2] for a more general, related result.

Extension of Ground Ring. When studying the structure of ideals in an extension K_n/K_0 over a group ring, one must choose a ring of coefficients. Does one study 'fine' structure – over $\mathfrak{O}_0[G]$ where the coefficients are integers in K_0. Does one study 'coarse' structure – over $\mathbb{Z}_2[G]$. We study a canonical intermediate structure – over $\mathfrak{O}_T[G]$ where the coefficients belong to the Witt ring of the residue class field. We determine this structure by listing generators and relations. Interestingly,

the coefficients in these relations always belong $\mathbb{Z}_2[G]$. Therefore $\mathfrak{O}_T[G]$-structure results, by extension of the ground ring, from $\mathbb{Z}_2[G]$-structure [CR90, §30B].

Realizable Modules. Let $\mathcal{S}_G$ denote the set of realizable indecomposable $\mathbb{Z}_2[G]$-modules: Those indecomposable $\mathbb{Z}_2[G]$-modules that appear in the decomposition of some ambiguous ideal in an extension N/K with $\mathrm{Gal}(N/K) \cong G$. Chinburg asked whether $\mathcal{S}_G$ could be infinite. In [Eld98], since $\mathcal{S}_{C_2\times C_2}$ is infinite, the answer was found to be *yes*. We determine here that although the set of indecomposable $\mathbb{Z}_2[C_8]$-modules is infinite, $\mathcal{S}_{C_8}$ is finite. The sequence $|\mathcal{S}_{C_{2^n}}|$, $n = 0, 1, 2, \dots$ begins

$$1, 3, 7, 23 \dots$$

1.3. **Organization of Paper.** Preliminary results are presented in §2, main results in §3. There are two appendixes. Appendix A lists all necessary indecomposable modules. Appendix B lists bases for our ideals.
Preliminary Results: In §2.1 we handle the special case when a ramification break number is even. In §2.2, we present a strategy for handling odd ramification numbers. To motivate our work in §3, we implement this strategy for $|G_1| = 2$ and 4, in §2.2.1 and §2.2.3 respectively. We conclude, in §2.3, with a reduction to totally ramified extensions.
Main Results: We begin in §3.1 with a brief outline and discussion. Then, we catalog ramification numbers and prove some technical lemmas in §3.2. All this sets the stage for our work in §3.3 determining the Galois structure of ideals in fully, though *unstably*, ramified C_8-extensions. This is our primary focus. Our work in §3.4 on *stably ramified* extensions is essentially contained in [Eld02].

2. Preliminary Results

We continue to use the notation of §1.1. Let K_0 be a finite extension of $\mathbb{Q}_2$ and K_n/K_0 be a cyclic extension of degree 2^n. Let σ generate $G = \mathrm{Gal}(K_n/K_0)$ and use subscripts to distinguish among subfields. So K_i denotes the fixed field of $\langle \sigma^{2^i} \rangle$, $\mathfrak{O}_i$ denotes the ring of integers of K_i and $\mathfrak{P}_i$ denotes the maximal ideal of $\mathfrak{O}_i$. Let v_i be the additive valuation in K_i, π_i its prime element, so that $v_i(\pi_i) = 1$. Let $\mathrm{Tr}_{i,j}$ denote the trace from K_i down to K_j. Recall the ramification filtration of G. Note $G_{-1} = G_0$ if and only if K_n/K_0 is fully ramified. Also since G is a 2-group and $[G_1 : G_0]$ is odd, $G_0 = G_1$. Furthermore since G is cyclic and G_i/G_{i-1} is elementary abelian for $i > 1$, there are $s = \log_2 |G_1|$ breaks in the filtration of G_1 [Ser79, p 67]. Let $b_1 < b_2 < \dots < b_s$ denote these break numbers. (The break numbers of G may include -1 as well.) It is a standard exercise to show that $b_1, \dots, b_s$ are *all* either odd or even [Ser79, Ex 3, p 71]. When they are even, we are in an extreme case, called *maximal ramification.* The general case, when they are odd, will be our primary concern.

2.1. **Even Ramification Numbers.** If $b_1, \dots, b_s$ are even, we use idempotent elements of the group algebra, $\mathbb{Q}_2[G]$, and Ullom's generalization of Noether's result [Ull70, Thm 1], to determine the structure of each ideal. In doing so, we rely upon two observations: (1) Idempotent elements in $\mathbb{Q}_2[G]$ that map an ideal into itself, decompose the ideal. (2) Modules over a principal ideal domain are free.

We illustrate this process in one case, leaving other cases to the reader. Suppose $|G| = 8$ and $|G_1| = 4$. So K_3/K_0 is only partially ramified and $s = 2$. From [Ser79, IV §2 Ex 3], $b_1 = 2e_0$ and $b_2 = 4e_0$. Using [Ser79, V §3], one finds that

$(1/2)(\sigma^4+1)\mathfrak{P}_3^i \subseteq \mathfrak{P}_3^i$. As a result, the idempotent $(\sigma^4+1)/2$ decomposes the ideal $\mathfrak{P}_3^i \cong \mathfrak{P}_2^{\lceil i/2\rceil} \oplus M_2$ with $(\sigma^4+1)M_2 = 0$. Meanwhile $(1/2)(\sigma^2+1)\mathfrak{P}_2^{\lceil i/2\rceil} \subseteq \mathfrak{P}_2^{\lceil i/2\rceil}$. So $\mathfrak{P}_2^{\lceil i/2\rceil}$ decomposes as well. This yields $\mathfrak{P}_3^i \cong \mathfrak{P}_1^{\lceil i/4\rceil} \oplus M_1 \oplus M_2$ with $(\sigma^4+1)M_2 = 0$ and $(\sigma^2+1)M_1 = 0$. Each M_i may be viewed as a module over $\mathfrak{O}_{T_K}[\sigma]/(\sigma^{2^i}+1)$, a principal ideal domain. So M_i is free over $\mathfrak{O}_{T_K}[\sigma]/(\sigma^{2^i}+1)$. Ullom's result provides a normal integral basis for $\mathfrak{P}_1^{\lceil i/4\rceil}$. Counting $\mathfrak{O}_T$-ranks, we find that

$$\mathfrak{P}_3^i \cong \frac{\mathfrak{O}_T[\sigma]}{(\sigma^2-1)}^{e_0} \oplus \frac{\mathfrak{O}_T[\sigma]}{(\sigma^2+1)}^{e_0} \oplus \frac{\mathfrak{O}_T[\sigma]}{(\sigma^4+1)}^{e_0}.$$

2.2. **Odd Ramification Numbers.** Henceforth the ramification numbers will be odd. In this context we will use the following technical result (with K_i/K_{i-1}).

Lemma 2.1. *Let k be a finite extension of $\mathbb{Q}_2$ and K/k be a ramified quadratic extension. Let e_k be the absolute ramification index of k. Assume that σ generates the Galois group and that the ramification number, $b < 2e_k$, is odd. Then*

(1) *$v_K((\sigma \pm 1)\alpha) = v_K(\alpha) + b$ for $v_K(\alpha)$ odd;*
(2) *if $\tau \in k$, there is a $\rho \in K$ such that $(\sigma+1)\rho = \tau$ and $v_K(\rho) = v_K(\tau) - b$;*
(3) *if $v_K(\alpha)$ is even and $(\sigma+1)\alpha = 0$, there is a $\theta \in K$ such that $\alpha = (\sigma-1)\theta$ and $v_K(\theta) = v_K(\alpha) - b$.*

Proof. These may be shown using [Ser79, V §3], as in [Eld98, Lem 3.12–14]. $\square$

Our *strategy* is based upon the following observations:
(1) Under wild ramification, Galois action 'shifts/increases' valuation (Lemma 2.1(1)). So an element may be used to 'construct' other elements with distinct valuation.
(2) Elements with distinct valuation may be used to construct bases. If the valuation map $v_n : K_n \to \mathbb{Z}$ is one–to–one on a subset $A \subseteq K_n$, while $v_n(A)$ is onto $\{i, i+1, \ldots, i+v_n(2)-1\}$; then A is a basis for $\mathfrak{P}_n^i$ over the integers in the maximal unramified subfield of K_n. If K_n/K_0 is fully ramified, this subfield is T.
The *strategy* is illustrated below. It is: Use *Galois Action* to *Create Bases*.

2.2.1. *First Ramification Group of Order Two.* Suppose that $|G_1| = 2$. To use *Observation* (1), we pick $\alpha \in K_n$ an element with $v_n(\alpha) = b_1$ (*e.g.* $\alpha = \pi_n^{b_1}$). Let $\alpha_m := \alpha \cdot \pi_0^m$. Since $v_n(\pi_0) = 2$, $v_n(\alpha_m) = b_1 + 2m$. Use Lemma 2.1 with K_n/K_{n-1}. So $v_n((\sigma^{2^{n-1}}+1)\alpha_m) = 2b_1 + 2m$. Since b_1 is odd, the valuations of α_m and $(\sigma^{2^{n-1}}+1)\alpha_m$ have opposite parity. The valuations for all m lie in one–to–one correspondence with $\mathbb{Z}$. Select those with valuation in $\{i, \ldots, i+v_n(2)-1\}$. Replace $\pi_0^{e_0}$ by 2 whenever possible. The result is

$$\begin{aligned}(2.1)\quad \mathcal{B} := &\left\{\alpha_m, (\sigma^{2^{n-1}}+1)\alpha_m : \lceil (i-b_1)/2\rceil \le m \le e_0 + \lceil i/2\rceil - b_1 - 1\right\}\\ &\cup \left\{(\sigma^{2^{n-1}}+1)\alpha_m, 2\alpha_m : \lceil i/2\rceil - b_1 \le m \le \lceil (i-b_1)/2\rceil - 1\right\}.\end{aligned}$$

Since K_{n-1}/K_0 is unramified, there is a root of unity ζ with $K_{n-1} = K_0(\zeta)$. The maximal unramified extension $\mathbb{Q}_2$ in K_n is $T(\zeta)$. By *Observation* (2), $\mathcal{B}$ is a basis for $\mathfrak{P}_n^i$ over $\mathfrak{O}_{T(\zeta)}$. Note that $\mathfrak{O}_{T(\zeta)}\cdot\alpha_m + \mathfrak{O}_{T(\zeta)}\cdot(\sigma^{2^{n-1}}+1)\alpha_m = \mathfrak{O}_{T(\zeta)}\cdot\alpha_m + \mathfrak{O}_{T(\zeta)}\cdot\sigma\alpha_m$ yields the group ring, $\mathfrak{O}_{T(\zeta)}[G_1]$, while $\mathfrak{O}_{T(\zeta)}\cdot(\sigma^{2^{n-1}}+1)\alpha_m + \mathfrak{O}_{T(\zeta)}\cdot 2\alpha_m = \mathfrak{O}_{T(\zeta)}\cdot(\sigma^{2^{n-1}}+1)\alpha_m + \mathfrak{O}_{T(\zeta)}\cdot(\sigma^{2^{n-1}}-1)\alpha_m$ yields the maximal order of $\mathfrak{O}_{T(\zeta)}[G_1]$. Restricting coefficients and counting leads to the $\mathfrak{O}_k[G_1]$-module structure of $\mathfrak{P}_n^i$, and to $\mathcal{M}_1(i, b_1)$ as in (1.1).

Next, we extend $\mathcal{B}$ to a basis upon which the action of the whole group can be followed. Since K_{n-1}/K_0 is unramified, there is a normal field basis for $\mathfrak{P}_{n-1}^j/\mathfrak{P}_{n-1}^{j+1}$ over $\mathfrak{O}_0/\mathfrak{P}_0$ (for each j). Of course, $[\mathfrak{O}_0/\mathfrak{P}_0 : \mathfrak{O}_T/\mathfrak{P}_T] = 1$. So $\mathfrak{P}_{n-1}^j/\mathfrak{P}_{n-1}^{j+1}$ has a normal field basis over $\mathfrak{O}_T/\mathfrak{P}_T$. For $j = b_1$, this means that there is an element $\mu \in \mathfrak{P}_{n-1}^{b_1}$ and basis $\mu, \sigma\mu, \dots, \sigma^{2^{n-1}-1}\mu$. Using Lemma 2.1(2), there is an $\alpha \in K_n$ with $v_n(\alpha) = b_1$ such that $(\sigma^{2^{n-1}} + 1)\alpha = \mu$. Then $\alpha, \sigma\alpha, \dots, \sigma^{2^{n-1}-1}\alpha$ is a normal field basis for $\mathfrak{P}_n^{b_1}/\mathfrak{P}_n^{b_1+1}$ over $\mathfrak{O}_T/\mathfrak{P}_T$. Since $\{\sigma^j(\sigma^{2^{n-1}} + 1)\alpha : j = 0, \dots, 2^{n-1} - 1\}$ is a basis for $\mathfrak{P}_{n-1}^{b_1}/\mathfrak{P}_{n-1}^{b_1+1}$, it is also a basis for $\mathfrak{P}_n^{2b_1}/\mathfrak{P}_n^{2b_1+1}$ over $\mathfrak{O}_T/\mathfrak{P}_T$. This together with the fact that $\{\sigma^j\alpha : j = 0, \dots, 2^{n-1} - 1\}$ is a basis for $\mathfrak{P}_n^{b_1}/\mathfrak{P}_n^{b_1+1}$ over $\mathfrak{O}_T/\mathfrak{P}_T$ leads to $\cup_{j=0}^{2^{n-1}-1}\sigma^j\mathcal{B}$ being a basis for $\mathfrak{P}_n^i$ over $\mathfrak{O}_T$, and $\mathfrak{P}_n^i \cong \mathfrak{O}_T[G] \otimes_{\mathbb{Z}_2[G_1]} \mathcal{M}_1(i, b_1)$ as $\mathfrak{O}_T[G]$-modules.

2.2.2. *An Application of Nakayama's Lemma.* In the previous section we were able to follow the Galois action from one basis element to another *explicitly.* This level of detail becomes overwhelming as we generalize to $|G_1| = 4, 8$. Fortunately, Nakayama's Lemma allows us to push some of these details into the background.

Lemma 2.2. *Let $\mathcal{A}$ be a $\mathfrak{O}_k[C_{2^n}]$-module (torsion-free over $\mathfrak{O}_k$) where $C_{2^n} = \langle\sigma\rangle$ and k denotes an unramified extension of $\mathbb{Q}_2$. Let H denote the subgroup of order 2, A^H the submodule fixed by H, and $\mathrm{Tr}_H A$ the image under the trace. Then $\mathrm{Tr}_H\mathcal{A}/\left((\sigma - 1)\mathrm{Tr}_H\mathcal{A} + 2\mathcal{A}^H\right)$ is free over $\mathfrak{O}_k/2\mathfrak{O}_k$. Suppose that $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathrm{Tr}_H\mathcal{B}$ is a basis for $\mathrm{Tr}_H\mathcal{A}/\left((\sigma - 1)\mathrm{Tr}_H\mathcal{A} + 2\mathcal{A}^H\right)$ then $\mathcal{B}$ can be extended to a $\mathfrak{O}_k[C_{2^n}]/\langle\mathrm{Tr}_H\rangle$-basis of $\mathcal{A}/\mathcal{A}^H$.*

Proof. Since $\mathcal{A}/\mathcal{A}^H$ is a module over the principal ideal domain $\mathfrak{O}_k[C_{2^n}]/\langle\mathrm{Tr}_H\rangle$, it is free. So $\mathcal{C} := \mathcal{A}/\mathcal{A}^H \cong (\mathfrak{O}_k[C_{2^n}]/\langle\mathrm{Tr}_H\rangle)^a$ for some exponent a. Now $\mathfrak{O}_k[C_{2^n}]/\langle\mathrm{Tr}_H\rangle$ is a local ring with maximal ideal $\langle\sigma - 1\rangle$ dividing 2. Therefore by Nakayama's Lemma any collection of elements in $\mathcal{A}$ that serves as a $\mathfrak{O}_k/2\mathfrak{O}_k$-basis for $\mathcal{C}/(\sigma-1)\mathcal{C}$ will serve as an $\mathfrak{O}_k[C_{2^n}]/\langle\mathrm{Tr}_H\rangle$-basis for $\mathcal{C}$. This leaves us to show that $\mathcal{B}$ can be extended to a $\mathfrak{O}_k/2\mathfrak{O}_k$-basis for the vector space $\mathcal{C}/(\sigma-1)\mathcal{C} = \mathcal{A}/(\mathcal{A}^H + (\sigma-1)\mathcal{A})$. But since $\mathrm{Tr}_H\mathcal{B}$ is a basis for $\mathrm{Tr}_H\mathcal{A}/\left((\sigma - 1)\mathrm{Tr}_H\mathcal{A} + 2\mathcal{A}^H\right)$, the elements of $\mathcal{B}$ are linearly independent in $\mathcal{A}/(\mathcal{A}^H + (\sigma - 1)\mathcal{A})$ and therefore span a subspace. $\square$

2.2.3. *First Ramification Group of Order Four.* Let $|G_1| = 4$. This case is important because it illustrates the utility of Lemma 2.2. (Recall that §2.2.1 and §2.2.3 are included in this paper to motivate considerations in §3.)
Step 1: Collect $|G_1|$ elements whose valuations are a complete set of residues modulo $|G_1|$. We begin with the elements used to determine the structure of ideals in K_{n-1} (from §2.2.1), namely α_m and $(\tilde{\sigma} + 1)\alpha_m \in K_{n-1}$ (replacing n by $n-1$, expressing $\sigma^{2^{n-2}}$ as $\tilde{\sigma}$). Note that the first ramification number of K_n/K_{n-2} is the (only) ramification number of K_{n-1}/K_{n-2} (use [Ser79, pg 64 Cor] or switch to upper ramification numbers [Ser79, IV §3]). So $v_n(\alpha_m) = 2v_{n-1}(\alpha_m) = 2b_1 + 4m$ and $v_n((\tilde{\sigma}+1)\alpha_m) = 4b_1+4m$. We have two elements of even valuation. To get elements with odd valuation, we apply Lemma 2.1(2). For each $X \in K_{n-1}$, Lemma 2.1(2) gives us a preimage $\overline{X} \in K_n$ (under the trace $\mathrm{Tr}_{n,n-1}$), a preimage that satisfies $v_n(\overline{X}) = 2v_{n-1}(X) - b_2$. So $\mathrm{Tr}_{n,n-1}\overline{X} = (\tilde{\sigma}^2 + 1)\overline{X} = X$. The integers $v_n(\alpha_m)$,

$v_n((\tilde\sigma+1)\alpha_m)$, $v_n(\overline{\alpha_m}) = 2b_1 - b_2 + 4m$, $v_n(\overline{(\tilde\sigma+1)\alpha_m}) = 4b_1 - b_2 + 4m$ are a complete set of residues modulo 4.

Step 2: Collect elements with valuation in $\{i, i+1, \dots, i+v_n(2)-1\}$*.* To organize this process we use Wyman's catalog of ramification numbers [Wym69]. If $b_1 \geq e_0$, the second ramification number is uniquely determined, $b_2 = b_1 + 2e_0$. If $b_1 < e_0$, then either $b_2 = 3b_1$, $b_2 = 4e_0 - b_1$, or $b_2 = b_1 + 4t$ for some t with $b_1 < 2t < 2e_0 - b_1$ [Wym69, Thm 32]. In any case, we have the bound,

$$2b_1 < b_2. \tag{2.2}$$

Now for a given m, list the infinitely many elements, α_{m+ke_0}, $(\tilde\sigma+1)\alpha_{m+ke_0}$, $\overline{\alpha_{m+ke_0}}$, $\overline{(\tilde\sigma+1)\alpha_{m+ke_0}}$, in terms of increasing valuation. Replace α_{m+ke_0} by $2^k\alpha_m$ and drop the subscripts m. So for $b_2 > 4e_0 - 2b_1$, beginning at $\overline{\alpha}$, we have:

$$\cdots \longrightarrow \overline{\alpha} \xrightarrow{1} 1/2(\tilde\sigma+1)\alpha \xrightarrow{2} \overline{(\tilde\sigma+1)\alpha} \xrightarrow{3} \alpha \xrightarrow{2} \overline{2\alpha} \longrightarrow \cdots$$

Each increase in valuation, denoted by $\xrightarrow{x}$, is justified as follows: For $x = 1$, the justification depends upon the case either $b_2 > 4e_0 - 2b_1$ or $b_2 < 4e_0 - 2b_1$. For $x = 2$, it is $b_2 < 4e_0$. For $x = 3$, it is (2.2). If $b_2 < 4e_0 - 2b_1$, the list is as follows:

$$\cdots \longrightarrow \overline{\alpha} \xrightarrow{4} \overline{(\tilde\sigma+1)\alpha} \xrightarrow{3} \alpha \xrightarrow{4} (\tilde\sigma+1)\alpha \xrightarrow{1} \overline{2\alpha} \longrightarrow \cdots$$

Note $x = 4$ is justified by $b_1 > 0$.

Now collect those elements with valuation in $\{i, \dots, i+v_n(2)-1\}$. This will provide us with an $\mathfrak{O}_{T(\zeta)}$-basis for $\mathfrak{P}_n^i$. Begin with the smallest m such that $i \leq v_n(\alpha_m)$. Note then that $v_n(\overline{2(\tilde\sigma+1)\alpha_m}) < i + v_n(2)$. Associated with this particular m are four elements in $\{i, \dots, i+v_n(2)-1\}$. They are listed in the first row of the table below. Consider this interval to be a 'window'. As we increase m, new elements appear (*e.g.* $2X$) – appearance coincides with disappearance (namely of X). Four elements are in 'view' always. There are four 'views' (four sets). We list the 'views' as rows under the appropriate heading.

$\mathcal{D}$: The $\mathfrak{O}_{T(\zeta)}$-basis for $\mathfrak{P}_n^i$.

	$A: \quad b_2 < 4e_0 - 2b_1$	$B: \quad b_2 > 4e_0 - 2b_1$
(1)	$\alpha \quad (\tilde\sigma+1)\alpha \quad \overline{2\alpha} \quad \overline{2(\tilde\sigma+1)\alpha}$	$\alpha \quad \overline{2\alpha} \quad (\tilde\sigma+1)\alpha \quad \overline{2(\tilde\sigma+1)\alpha}$
(2)	$\overline{(\tilde\sigma+1)\alpha} \quad \alpha \quad (\tilde\sigma+1)\alpha \quad \overline{2\alpha}$	$\overline{(\tilde\sigma+1)\alpha} \quad \alpha \quad \overline{2\alpha} \quad (\tilde\sigma+1)\alpha$
(3)	$\overline{\alpha} \quad \overline{(\tilde\sigma+1)\alpha} \quad \alpha \quad (\tilde\sigma+1)\alpha$	$1/2(\tilde\sigma+1)\alpha \quad \overline{(\tilde\sigma+1)\alpha} \quad \alpha \quad \overline{2\alpha}$
(4)	$1/2(\tilde\sigma+1)\alpha \quad \overline{\alpha} \quad \overline{(\tilde\sigma+1)\alpha} \quad \alpha$	$\overline{\alpha} \quad 1/2(\tilde\sigma+1)\alpha \quad \overline{(\tilde\sigma+1)\alpha} \quad \alpha$

Should we need to determine the subscripts (associated with a particular 'view'), we can easily do so: For example the four elements listed in $A(1)$ and $B(1)$, appear for m with $i \leq v_n(\alpha_m)$ and $v_n(\overline{2(\tilde\sigma+1)\alpha_m}) \leq i + 4e_0 - 1$. In other words, $\lceil (i-2b_1)/4 \rceil \leq m \leq \lceil (i+b_2)/4 \rceil - b_1 - 1$.

Step 3: Identify a basis for the quotient module $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2 \rceil}$*, and determine the precise image of each basis element under the trace* $Tr_{n,n-1}$ *(in terms of the basis for* $\mathfrak{P}_{n-1}^{\lceil i/2 \rceil}$*).* Observe that $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2 \rceil}$ is, in a natural way, free over the principal ideal domain $\mathfrak{O}_{T(\zeta)}[G]/\langle \tilde\sigma^2 + 1 \rangle$. We begin by identifying those elements listed in $\mathcal{D}$, the $\mathfrak{O}_{T(\zeta)}$-basis from *Step 2*, that can serve as a $\mathfrak{O}_{T(\zeta)}[G]/\langle \tilde\sigma^2+1 \rangle$-basis. Take $\mathcal{D}$

and partition it into two sets. Let $\overline{\mathcal{D}}$ contain those elements $\overline{X}$ with a bar. Let $\mathcal{D}_0$ contain those elements X without a bar. So $\overline{\mathcal{D}}$ is an $\mathfrak{O}_{T(\zeta)}$-basis for $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$, and $\mathcal{D}_0$ is an $\mathfrak{O}_{T(\zeta)}$-basis for $\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$. If we knew which elements from $\overline{\mathcal{D}}$ provide us with $\mathfrak{O}_{T(\zeta)}[G]/\langle\tilde{\sigma}^2+1\rangle$-basis for $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$ we would be done, as it is easy to express the image (under the trace $\mathrm{Tr}_{n,n-1}$) of each element in $\overline{\mathcal{D}}$ in terms of elements of $\mathcal{D}_0$ (there is a one–to–one correspondence).

Before we proceed further, note the following. We may assume without loss of generality that for $\overline{X}\in\overline{\mathcal{D}}$, $\mathrm{Tr}_{n,n-1}\overline{X}\neq 0$ *if and only if* $\overline{X}$ appears together with X (for the same subscript m) in $\mathcal{D}$. Clearly if $\overline{X}$ and X appear together, then $\mathrm{Tr}_{n,n-1}\overline{X}=X\neq 0$. However when $\overline{2X}$ and X appear together, after a change of basis, we may assume that $\mathrm{Tr}_{n,n-1}\overline{2X}=0$. The reason for this is as follows: We can change an element of $\overline{\mathcal{D}}$ by adding an element from $\mathcal{D}_0$ and still have a $\mathfrak{O}_{T(\zeta)}$-basis for $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$. So whenever $\overline{2X}$ and X appear together, replace $\overline{2X}$ with $\overline{2X}-X$. Note $\mathrm{Tr}_{n,n-1}(\overline{2X}-X)=0$. If we perform this change throughout our basis, but relabel $\overline{2X}-X$ as $\overline{2X}$, then we may continue to use the lists, $A(1)$–$A(4)$ and $B(1)$–$B(4)$, but assume that $\mathrm{Tr}_{n,n-1}\overline{2X}=0$ if $\overline{2X}$ appears together with X.

Our next step will be to provide an $\mathfrak{O}_{T(\zeta)}[G]/\langle\tilde{\sigma}^2+1\rangle$-basis for $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$. Consider those rows with an $\overline{X}$ such that $\mathrm{Tr}_{n,n-1}\overline{X}\neq 0$ (namely $A(2)$, $A(3)$, $A(4)$, $B(2)$, $B(4)$). Let $\mathcal{S}\subseteq\overline{\mathcal{D}}$ denote the set of *left–most* $\overline{X}$ associated with those rows. So, for example, if $b_2+2b_1<4e_0$, then $\mathcal{S}$ is made up of the $\overline{(\tilde{\sigma}+1)\alpha_m}$ from $A(2)$, and the $\overline{\alpha_m}$ from $A(3)$ and $A(4)$. Verify that $\mathrm{Tr}_{n,n-1}\mathcal{S}$ is a $\mathfrak{O}_{T(\zeta)}/2\mathfrak{O}_{T(\zeta)}$-basis for $\mathrm{Tr}_{n,n-1}\mathfrak{P}_n^i/((\tilde{\sigma}-1)\mathrm{Tr}_{n,n-1}\mathfrak{P}_n^i+2\mathfrak{P}_{n-1}^{\lceil i/2\rceil})$ (observe that $\mathrm{Tr}_{n,n-1}\mathcal{S}$ generates $\mathrm{Tr}_{n,n-1}\mathfrak{P}_n^i/2\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$ over $\mathfrak{O}_{T(\zeta)}/2\mathfrak{O}_{T(\zeta)}[G]$). Now use Lemma 2.2 to extend $\mathcal{S}$ to $\mathcal{S}'$, an $\mathfrak{O}_{T(\zeta)}[G]/\langle\tilde{\sigma}^2+1\rangle$-basis for $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$. Since $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$ has rank e_0 over $\mathfrak{O}_{T(\zeta)}[G]/\langle\tilde{\sigma}^2+1\rangle$, we have $|\mathcal{S}'|=e_0$.

This $\mathfrak{O}_{T(\zeta)}[G]/\langle\tilde{\sigma}^2+1\rangle$-basis, $\mathcal{S}'$, possesses two important properties. First, it contains $\mathcal{S}$. Second, without loss of generality we may assume that the elements in $\mathcal{S}'-\mathcal{S}$ are killed by the trace $\mathrm{Tr}_{n,n-1}$. These two properties are shared with another set: The set of **all** *left–most* $\overline{X}$ (an $\overline{X}$ for every value of m). Clearly the set of all *left–most* $\overline{X}$ contains $\mathcal{S}$. Moreover, by an earlier assumption, the compliment of $\mathcal{S}$ in the set of all *left–most* $\overline{X}$ is mapped to zero under the trace. And so, because the sets have the same cardinality (namely e_0), we can identify them. Without loss of generality, assume that $\mathcal{S}'$ *is* the set of all *left–most* $\overline{X}$. This allows us to use the lists, $A(1)$–$A(4)$ and $B(1)$–$B(4)$, in the 'book-keeping' necessary for determining the Galois module structure below.

At this point, we know that $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$ is free over $\mathfrak{O}_{T(\zeta)}[G]/\langle\tilde{\sigma}^2+1\rangle$. Indeed, $\mathcal{S}'$ (the set of all *left–most* $\overline{X}$) provides us a $\mathfrak{O}_{T(\zeta)}[G]/\langle\tilde{\sigma}^2+1\rangle$-basis for $\mathfrak{P}_n^i/\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$. Of course, the $\mathfrak{O}_{T(\zeta)}[G]$-structure of $\mathfrak{P}_{n-1}^{\lceil i/2\rceil}$ is known from §2.2.1 (and can be read off of $\mathcal{D}_0$). So a description of the image of $\mathcal{S}'$ under $\tilde{\sigma}^2+1$ in terms of $\mathcal{D}_0$ will determine the Galois module structure. See [CR90, §8]. *The Result*: For each m associated with $A(1)$ or $B(1)$ we decompose off an $\mathfrak{O}_{T(\zeta)}[G_1]$-summand of $\mathfrak{O}_{T(\zeta)}\otimes_{\mathbb{Z}_2}\mathcal{I}$, for $A(2)$ or $B(2)$ we get an $\mathfrak{O}_{T(\zeta)}\otimes_{\mathbb{Z}_2}\mathcal{H}$, for $A(3)$ we find the group ring, $\mathfrak{O}_{T(\zeta)}[G_1]\cong\mathfrak{O}_{T(\zeta)}\otimes_{\mathbb{Z}_2}\mathcal{G}$. But, for $B(3)$ we decompose off the maximal order of $\mathfrak{O}_{T(\zeta)}[G_1]$, $\mathfrak{O}_{T(\zeta)}\otimes_{\mathbb{Z}_2}\mathcal{M}$. For $A(4)$ and $B(4)$ there is $\mathfrak{O}_{T(\zeta)}\otimes_{\mathbb{Z}_2}\mathcal{L}$. All this and

counting determines the $\mathfrak{O}_{T(\zeta)}[G_1]$-module structure of $\mathfrak{P}_n^i$ from which the $\mathfrak{O}_k[G_1]$-module structure can be inferred. It also determines the module $\mathcal{M}_2(i, b_1, b_2)$. To determine the $\mathfrak{O}_T[G]$-module structure (from which the $\mathfrak{O}_k[G]$-module structure can be inferred), we need to take our $\mathfrak{O}_{T(\zeta)}$-bases for $\mathfrak{P}_n^i$ and create $\mathfrak{O}_T$-bases.

2.3. **Partially Ramified Extensions.** Let T_i denote the maximal unramified extension of $\mathbb{Q}_2$ contained in K_i. So $T(\zeta)$ of the previous section can be expressed at T_n, while $T = T_0$. Recall the steps in §2.2.1. We first determined a $\mathfrak{O}_{T_n}$-basis $\mathcal{B}$ for $\mathfrak{P}_n^i$, one upon which the action of G_1 could be explicitly followed. Then noting that we can identify G/G_1 with the Galois group for T_n/T_0, we extended $\mathcal{B}$ to an $\mathfrak{O}_{T_0}$-basis for $\mathfrak{P}_n^i$. This time the action of every element in the Galois group G could be followed. What were the important ingredients in this process? It was important that the elements of $\mathcal{B}$ lay in one–to–one correspondence, via valuation, with the integers $i, \dots, i + v_n(2) - 1$. Using this fact and the fact that for each t, $\mathfrak{P}_n^t/\mathfrak{P}_n^{t+1}$ had a normal field basis over $\mathfrak{O}_{T_0}/\mathfrak{P}_{T_0}$, we were able to make an $\mathfrak{O}_T$-basis for $\mathfrak{P}_n^i$, namely $\mathcal{B}' = \cup_{\sigma^i \in G/G_1} \sigma^i \mathcal{B}$. At that point we were done. The $\mathfrak{O}_T[G]$-structure could simply be read off of this basis. This is not the case when $|G_1| = 4$. Nor is it the case when $|G_1| = 8$. At this point we still need to change our basis and use Nakayama's Lemma, if only to determine $\mathfrak{O}_T[G_1]$-structure. We leave it to the reader to check that this process of basis change 'commutes' with the process of extending our $\mathfrak{O}_{T_n}$-basis to an $\mathfrak{O}_{T_0}$-basis. Simply follow the argument using elements of the form $\sigma^t \alpha_m$, $\sigma^t(\tilde{\sigma} + 1)\alpha_m, \dots$ with $t = 0, \dots 2^{[G:G_1]-1}$ instead of elements of the form α_m, $(\tilde{\sigma} + 1)\alpha_m, \dots$. As a consequence, the problem of determining the $\mathfrak{O}_T[G]$-module structure reduces to the problem of determining the $\mathfrak{O}_{T_n}[G_1]$-module structure.

3. Fully Ramified Cyclic Extensions of Degree Eight

We consider fully ramified extensions K_3/K_0 with odd ramification numbers.

3.1. **Outline.** Our discussion here is focused on the *unstably ramified* case, $b_1 < e_0$. (The *stably ramified* case will be addressed separately in §3.4.) Recall *Step 1* of §2.2.3 (in reference to K_2/K_0). But first note that the first two ramification numbers of K_3/K_0 are the (only) two ramification numbers of K_2/K_0 [Ser79, pg 64 Cor]. We began with two elements, namely $\alpha, (\sigma + 1)\alpha$ in the subfield K_1. (The Galois relationship between them was explicit.) Then we created $\overline{\alpha}, \overline{(\sigma + 1)\alpha} \in K_2$, preimages under the trace $\mathrm{Tr}_{2,1}$. In this section, we will start with these four elements from K_2 and use Lemma 2.1(2) to find further preimages: of $\alpha, (\sigma+1)\alpha, \overline{\alpha}, \overline{(\sigma + 1)\alpha}$ under $\mathrm{Tr}_{3,2}$. To avoid confusion (confusion resulting from additional bars denoting a preimage under $\mathrm{Tr}_{3,2}$), we relabel. Let $\alpha := \overline{\alpha}$ and let $\rho := \overline{(\sigma + 1)\alpha}$. So the four elements in K_2 are labeled $\alpha, (\sigma^2 + 1)\alpha, \rho, (\sigma + 1)(\sigma^2 + 1)\alpha$ (instead of $\overline{\alpha}, \alpha, \overline{(\sigma + 1)\alpha}, (\sigma + 1)\alpha$ respectively). The eight resulting elements (four from K_2 along with their preimages) lie in one–to–one correspondence with the residues modulo 8.

We would have accomplished all that was accomplished in *Step 1* from §2.2.3 if we knew the Galois relationships among $\alpha, (\sigma^2 + 1)\alpha, \rho, (\sigma + 1)(\sigma^2 + 1)\alpha$ explicitly. We need an explicit relationship between α and ρ. This is accomplished in §3.2.2 through a list of technical results – generalizations of Lemma 2.1. Note that ρ is an 'approximation' to $(\sigma + 1)\alpha$ – they have the same image under the trace $\mathrm{Tr}_{2,1}$. Our results describe their difference, the 'error' in this 'approximation'.

As a prerequisite for the technical results of §3.2.2, and in preparation for the analog of *Step 2* from §2.2.3 we use a result of Fontaine to provide a catalog of ramification numbers in §3.2.1. We are then ready for *Step 2*: First we order the eight elements (that we inherit from *Step 1*) in terms of increasing valuation. This is accomplished in §3.3. There are eight orderings – eight cases. The result is eight different bases, listed as $A - H$ (as opposed to just two in $\mathcal{D}$ from §2.2.3). For the convenience of the reader, they are listed in Appendix B.

We are now ready for the analog of *Step 3* from §2.2.3. We are ready to determine those elements in each $\mathfrak{O}_T$-basis that serve as an $\mathfrak{O}_T[G]/\langle \mathrm{Tr}_{3,2}\rangle$-basis, $\mathcal{S}$, for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$. We will then be able to describe the image, $\mathrm{Tr}_{3,2}S$, in terms of our $\mathfrak{O}_T$-basis for $\mathfrak{P}_2^{\lceil i/2\rceil}$ (or more to the point, explicitly in terms of $\mathfrak{O}_T[G]$-generators for $\mathfrak{P}_2^{\lceil i/2\rceil}$). To do all this we will need, as in §2.2.3, to perform certain basis changes. The processes are similar, but there are a few very important differences. For the convenience of the reader, the results of this step are summarized in §3.3.1. The steps are then spelled out in §3.3.2 – §3.3.5. The structure of $\mathfrak{M}_3(i, b_1, b_2, b_3)$ (given in Tables 1 and 2) can then be read off of the bases in Appendix B. Note however, that we still need to determine the structure under $b_1 \geq e_0$ (part of Case A). This situation is addressed in §3.3.4.

3.2. **Preliminary Results.** We catalog the ramification triples and generalize Lemma 2.1, describing the difference $\rho - (\sigma+1)\alpha$.

3.2.1. *Ramification Triples.* There is *stability* and *instability*.

Theorem 3.1 ([Fon71, Prop 4.3])**.**
Stability:

$$b_1 \geq e_0 \Rightarrow b_2 = b_1 + 2e_0, \qquad \textit{and} \qquad b_1 + b_2 \geq 2e_0 \Rightarrow b_3 = b_2 + 4e_0.$$

Instability:

$$b_1 < e_0 \Rightarrow 3b_1 \leq b_2 \leq 4e_0 - b_1, \qquad b_1 + b_2 < 2e_0 \Rightarrow 3b_2 + 2b_1 \leq b_3 \leq 8e_0 - b_2 - 2b_1.$$

In particular, when $b_1 < e_0$, either $b_2 = 3b_1$, $b_2 = 4e_0 - b_1$, or $b_2 = b_1 + 4t$ for $b_1 < 2t < 2e_0 - b_1$, while if $b_1 + b_2 < 2e_0$, then either $b_2 = 3b_2 + 2b_1$, $b_2 = 8e_0 - b_2 - 2b_1$, or $b_3 = 8s - b_2 + 2b_1$ for $b_2 < 2s < 2e_0 - b_1$.

Plot these ramification triples (b_1, b_2, b_3) in $\Re^3$, and project this plot to the first two coordinates, $(x, y, z) \to (x, y, 0)$, thus creating Figure 1 (next page). This projection is partly a line: for $b_1 \geq e_0$, each point (b_1, b_2) is restricted to $b_2 = b_1 + 2e_0$. It is partly a triangular region: for $b_1 < e_0$, each point (b_1, b_2) is bound between the lines $b_2 = 3b_1$ and $b_2 = 4e_0 - b_1$. The significance of the regions $A, B, C, \ldots$ will be explained later. Note that for points, (b_1, b_2), above the line $b_2 = -b_1 + 2e_0$, the plot of the (b_1, b_2, b_3) in $\Re^3$ will be a plane – b_3 is uniquely determined.

In Figure 2 we have plotted a slice, at a particular value of b_1, through our plot of ramification triples in $\Re^3$. Part of this slice is a line – when b_3 is uniquely determined. Thus the line from $(2e_0 - b_1, 6e_0 - b_1)$ to $(4e_0 - b_1, 8e_0 - b_1)$. Indeed, as drawn, Figure 2 implicitly assumes that the slice was taken at b_1 for $b_1 < e_0/2$. Otherwise there would be no triangular region. Observe that in Figure 1, the lines $b_2 = 2e_0 - b_1$ and $b_2 = 3b_1$ intersect at $b_1 = e_0/2$. If $b_1 \geq e_0/2$, the third ramification number is uniquely determined by b_2. The triangular region bound by the lines $b_2 = 3b_1$, $b_3 = 3b_2 + 2b_1$ and $b_3 = 8e_0 - b_2 - 2b_1$ exists only for $b_1 < e_0/2$.

Because the ramification numbers are odd, the triangular part of Figure 1 can be partitioned as follows:

A. $4e_0 - 4b_1/3 < b_2$
B. $4e_0 - 2b_1 < b_2 < 4e_0 - 4b_1/3$
C. $4e_0 - 4b_1 < b_2 < 4e_0 - 2b_1$ and $b_2 > (4e_0 + 4b_1)/3$
D. $4e_0 - 4b_1 < b_2 < 4e_0 - 2b_1$ and $b_2 < (4e_0 + 4b_1)/3$
E. $2e_0 - b_1 \leq b_2 < 4e_0 - 4b_1$ and $b_2 > (4e_0 + 4b_1)/3$
F. $2e_0 - b_1 \leq b_2 < 4e_0 - 4b_1$ and $b_2 < (4e_0 + 4b_1)/3$

Fig. 1

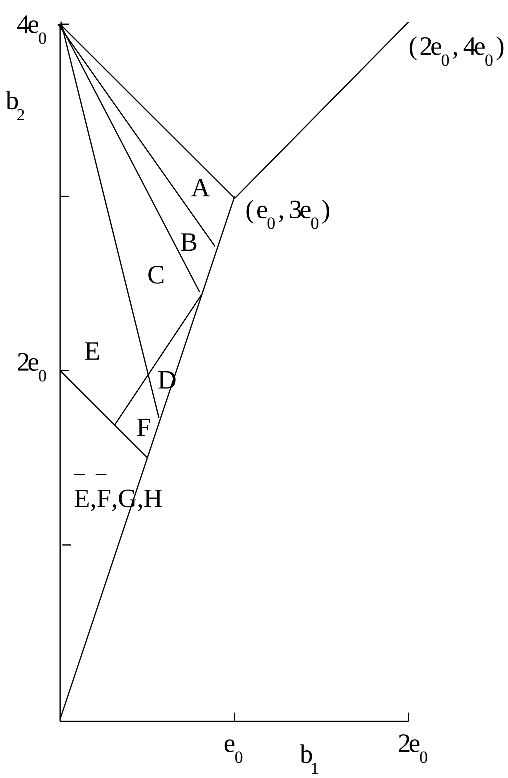
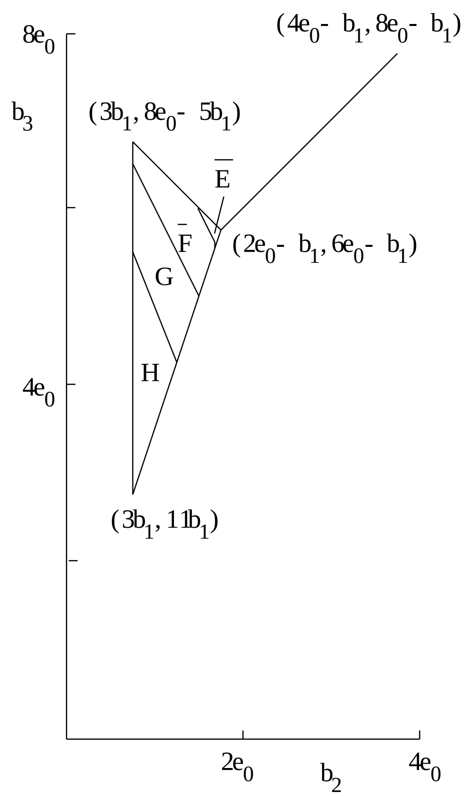


Fig. 2



Assuming that $b_1 < e_0/2$, there is a triangular region in Figure 2. This can be partitioned into the following cases:

$\overline{E}$. $8e_0 + 4b_1 - 2b_2 < b_3$
$\overline{F}$. $8e_0 - 2b_2 < b_3 < 8e_0 + 4b_1 - 2b_2$
G. $8e_0 - 4b_1 - 2b_2 < b_3 < 8e_0 - 2b_2$
H. $b_3 < 8e_0 - 4b_1 - 2b_2$

Note that if $b_1 > 8e_0/17$, region G is empty; if $b_1 > 8e_0/21$, region H is empty; if $b_1 > 8e_0/28$, region $\overline{E}$ is empty. So as drawn, we have assumed that $b_1 < 2e_0/7$. If however the slice were taken for a value $8e_0/17 < b_1 < 8e_0/16$, note that the triangular region would consist of only one case, namely $\overline{F}$. The relationship between E, F and $\overline{E}$, $\overline{F}$ will be explained in §3.3.

3.2.2. *Technical Lemmas.* The difference $\rho - (\sigma+1)\alpha$ depends upon ramification. *Unstable Ramification.* Assume that $b_1 < e_0$. These results may be thought of as consequences of indirect 'routes' from α to ρ. For example, we may begin with $\alpha \in K_2$, create $(\sigma^2+1)\alpha$, then $(\sigma+1)(\sigma^2+1)\alpha$ and let ρ be the inverse image of $(\sigma+1)(\sigma^2+1)\alpha$ under $\mathrm{Tr}_{2,1}$. This results in an expression for the $\rho - (\sigma+1)\alpha$.

Lemma 3.2. *If $b_2 \equiv b_1 \bmod 4$ (equivalently $3b_1 < b_2 < 4e_0 - b_1$), let $t = (b_2 - b_1)/4$. There are elements $\alpha_m \in K_2$ with $v_2(\alpha_m) = b_2 + 4m$, such that*

$$\rho_m = (\sigma+1)\alpha_m + (\sigma^2 \pm 1)\alpha_{m-t}$$

has valuation $v_2(\rho_m) = b_2 + 2b_1 + 4m$. The '+' or '−' depends on our needs.

Proof. Let $\alpha \in K_2$ with valuation, $v_2(\alpha) \equiv b_2 \bmod 4$. Using Lemma 2.1, $v_2((\sigma+1)\alpha) = v_2(\alpha) + b_1$, $v_2((\sigma^2+1)\alpha) = v_2(\alpha) + b_2$. Since $(\sigma^2+1)\alpha \in K_1$ and $v_1((\sigma^2+1)\alpha) = (v_2(\alpha)+b_2)/2 \equiv b_2 \bmod 2$, $v_1((\sigma+1)(\sigma^2+1)\alpha) = (v_2(\alpha)+b_2)/2 + b_1$. Using Lemma 2.1(2), there is a $\rho \in K_2$ with $v_2(\rho) = v_2(\alpha) + 2b_1$ such that $(\sigma^2+1)\rho = (\sigma+1)(\sigma^2+1)\alpha$. Since $(\sigma^2+1)\,[\rho - (\sigma+1)\alpha] = 0$. Using Lem 2.1(3), there is a $\theta \in K_2$ with $v_2(\theta) = (v_2(\alpha) - b_2) + b_1$ and $\rho = (\sigma+1)\alpha + (\sigma^2-1)\theta$. Since $b_1 < e_0$, $v_2(2\theta) > v_2(\rho)$. We may replace ρ by $\rho' := \rho + 2\theta$ (they have the same valuation), and get $\rho' = (\sigma+1)\alpha + (\sigma^2+1)\theta$. Once α_m is chosen, we let $\alpha_{m-t} := \theta$. □

Lemma 3.3. *If $b_2 \equiv -b_1 \bmod 4$ (equivalently $b_2 = 3b_1$ or $b_2 = 4e_0 - b_1$), let $s := (b_2 + b_1)/4$. There are elements $\alpha_m \in K_2$ with $v_2(\alpha_m) = b_2 + 4m$, such that*

$$\rho_m = (\sigma+1)\alpha_m + (\sigma+1)(\sigma^2+1)\alpha_{m-s}$$

has valuation, $v_2(\rho_m) = 2b_2 - b_1 + 4m$. Note if $b_2 = 3b_1$, $v_2(\rho_m) = b_2 + 2b_1 + 4m$.

Proof. There is a $\tau \in K_0$ with $v_0(\tau) = (b_2 - b_1)/2$. Using Lemma 2.1(2), let $\rho \in K_2$ with $v_2(\rho) = b_2 - 2b_1$ such that $(\sigma^2+1)\rho = \tau$. Clearly $(\sigma^2+1)\cdot(\sigma-1)\rho = 0$, so there is a $\theta \in K_2$ with $v_2(\theta) = -b_1$ such that $(\sigma-1)\rho = (\sigma^2-1)\theta$. Since $(\sigma-1)\cdot[\rho - (\sigma+1)\theta] = 0$, $\tau' := \rho - (\sigma+1)\theta$ is a unit in K_0. Let $\rho' = \rho/\tau'$ and $\theta' = \theta/\tau'$, so $1 = \rho' - (\sigma+1)\theta'$. Now let $\beta = (\sigma+1)(\sigma^2+1)\theta'$. Clearly $(\sigma^2+1)\theta' \in K_1$ and $v_1((\sigma^2+1)\theta') = (b_2 - b_1)/2$ is odd. Therefore $v_2(\beta) = b_2 + b_1$. Replacing 1 with the expression, $(\sigma+1)(\sigma^2+1)(\theta'/\beta)$, yields

$$\rho' = (\sigma+1)\theta' + (\sigma+1)(\sigma^2+1)(\theta'/\beta). \tag{3.1}$$

By choosing $\tau \in K_0$ with other valuations, the result follows. □

Unfortunately, if $b_2 = 4e_0 - b_1$ then $s = e_0$ (valuation can not distinguish between $\alpha_m/2$ and α_{m-s}). To avoid this confusion, we include the following.

Lemma 3.4. *Let $b_2 = 4e_0 - b_1$. There are $\alpha_m \in K_2$ with $v_2(\alpha_m) = b_2 + 4m$, so*

$$\rho_m := (\sigma+1)\alpha_m - \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha_m + \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$$

has valuation, $v_2(\rho_m) = 2b_2 - b_1 + 4m$.

Proof. From (3.1) we have $\rho' = (\sigma+1)\theta' + (\sigma+1)(\sigma^2+1)(\theta'/\beta)$. Apply $(\sigma^2+1)/\beta$ to both sides. So $(\sigma^2+1)(\rho'/\beta) = 1 + 2/\beta$. Since $v_2((\sigma^2+1)\rho') = 8e_0 - 4b_1$ and $v_2(\beta) = 4e_0$, then $v_0(1+2/\beta) = e_0 - b_1$. Replace θ'/β with $(1/2)\cdot[-\theta' + \theta'(1+2/\beta)]$, and distribute $(\sigma+1)(\sigma^2+1)$. □

Remark 3.5. Note $(\sigma-1)\rho_m = (\sigma^2-1)\alpha_m$ and $(\sigma^2+1)\rho_m = (\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$, using Lemma 3.4. Apparently, ρ_m is 'torn' between α_m and $\alpha_{m+e_0-b_1}$. We chose to emphasize ρ_m's tie to α_m. If we relabel $\rho_{m-e_0+b_1}$ as ρ_m (keep the α_m the same), Lemma 3.4 reads

$$\rho_m := (\sigma+1)\alpha_{m-e_0+b_1} - \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha_{m-e_0+b_1} + \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha_m$$

has valuation, $v_2(\rho_m) = b_2 + 2b_1 + 4m$ – thus tying ρ_m to $(1/2)(\sigma+1)(\sigma^2+1)\alpha_m$. This valuation of ρ_m is as in Lemmas 3.2 and 3.3 (for $b_2 = 3b_1$).

Stably Ramified Extensions. Assume that $b_1 \geq e_0$. The results may be seen as direct routes from α to ρ. We create ρ immediately from $(\sigma+1)\alpha \in K_2$. For discussion and generalization, see [Eld02].

Lemma 3.6. *Let $b_1 > e_0$. For every odd integer, a, there are elements $\alpha, \rho \in K_2$ with $v_2(\alpha) = a$, $v_2(\rho) = a + (b_2 - b_1)$. such that*

$$(\sigma+1)\alpha - \rho = \mu \in K_1,$$

with $v_2(\mu) = v_2(\alpha) + b_1$. Furthermore $\mu \in K_0$ for $v_2(\mu) = v_2(\alpha) + b_1 \equiv 0 \bmod 4$.

Proof. Since $v_2((\sigma+1)\alpha) = v_2(\alpha) + b_1$ is even, we may express $(\sigma+1)\alpha$ as a sum $\mu + \rho$ with $\mu \in K_1$, $\rho \in K_2$, $v_2(\mu) = v_2(\alpha) + b_1$ and odd $v_2(\rho)$. Apply $(\sigma - 1)$. So $(\sigma^2-1)\alpha = (\sigma-1)\mu + (\sigma-1)\rho$. Since $b_2 = b_1 + 2e_0 < 3b_1$, $v_2((\sigma^2-1)\alpha) = v_2(\alpha) + b_2 < v_2(\alpha) + 3b_1 \leq v_2((\sigma-1)\mu)$. So $v_2((\sigma^2-1)\alpha) = v_2((\sigma-1)\rho)$ and $v_2(\rho) = v_2(\alpha) + (b_2 - b_1)$. If $v_2(\mu) \equiv 0 \bmod 4$, we may choose α so that $\mu \in K_0$. Pick a $\mu^* \in K_0$ with $v_2(\mu^*) = v_2(\mu)$. Relabel α as α_0. Choose $\alpha_i \in K_2$ with $v_2(\alpha_i) = v_2(\alpha_0) + 2i$. As before, generate μ_i and ρ_i with $\alpha_i = \mu_i + \rho_i$. Clearly $\mu^* = \sum_{i=0}^{\infty} a_i\mu_i$ for some units $a_i \in K_0$. Let $\alpha^* = \sum_{i=0}^{\infty} a_i\alpha_i$ and $\rho^* = \sum_{i=0}^{\infty} a_i\rho_i$. □

Lemma 3.7. *Let $b_1 = e_0$ be odd. For every odd integer, a, there are elements $\alpha, \rho \in K_2$ with $v_2(\alpha) = a$, $v_2(\rho) = a + (b_2 - b_1)$ such that*

$$(\sigma-1)\alpha - \rho = \mu_1 \in K_1 \text{ if } a \equiv e_0 \bmod 4,$$
$$(\sigma+1)\alpha - \rho = \mu_0 \in K_0 \text{ if } a \equiv 3e_0 \bmod 4.$$

with $v_2(\mu_i) = v_2(\alpha) + b_1$.

Proof. Let $\tau \in K_0$ be a unit. From Lemma 2.1(2), there is a $\rho \in K_2$ with $v_2(\rho) = -b_2$ and $(\sigma^2+1)\rho = \tau$. So $(\sigma^2+1)\cdot(\sigma-1)\rho = 0$. Use Lemma 2.1(3) to find $\theta \in K_2$ with $v_2(\theta) = b_1 - 2b_2$ and $(\sigma^2-1)\theta = (\sigma-1)\rho$. For $a \equiv e_0 \bmod 4$, we may assume that $\alpha = \rho\pi_0^m$ for some m. Let $\mu_1 = (\sigma^2+1)\theta\pi_0^m \in K_1$ and $\rho = -2\theta\pi_0^m \in K_2$. The statement follows. For $a \equiv 3e_0 \bmod 4$, $(\sigma^2-1)\theta = (\sigma-1)\rho$ can be interpreted to mean that $\rho - (\sigma+1)\theta \in K_0$. Multiplying by an appropriate power of π_0, we let $\alpha = \theta\pi_0^m$, $\mu_0 = -(\rho - (\sigma+1)\theta)\pi_0^m \in K_0$ and $\rho = \rho\pi_0^m \in K_2$. □

3.3. **The Galois module structure under *unstable* ramification.** Assume $b_1 < e_0$. First we determine the $\mathfrak{O}_T$-bases in Appendix B. From Lemmas 3.2, 3.3, 3.4 we have $\alpha_m, \rho_m, (\sigma^2+1)\alpha_m, (\sigma+1)(\sigma^2+1)\alpha_m \in K_2$, with valuations (measured in v_2) for every residue class modulo 4. Recall $v_2(\alpha_m) = b_2 + 4m$, $v_2((\sigma^2+1)\alpha_m) = 2b_2 + 4m$, $v_2((\sigma+1)(\sigma^2+1)\alpha_m) = 2b_2 + 2b_1 + 4m$ and $v_2(\rho_m) = 8e_0 - 3b_1 + 4m$ if $b_2 = 4e_0 - b_1$, otherwise $v_2(\rho_m) = b_2 + 2b_1 + 4m$. Using Lemma 2.1(2) we determine elements $\overline{\alpha_m}$, $\overline{\rho_m}$, $\overline{(\sigma^2+1)\alpha_m}$, $\overline{(\sigma+1)(\sigma^2+1)\alpha_m} \in K_3$, with $(\sigma^4+1)\overline{X} = X$ and $v_3(\overline{X}) = 2v_2(X) - b_3$. These eight elements have valuations (measured in v_3) in

one–to–one correspondence with the residue classes modulo 8. By varying m, it is possible to choose those with valuation $i \le v_3(x) < 8e_0 + i$.

To organize this process, we list these elements in terms of increasing valuation. There are eight orderings – eight cases. In each case X (or $\overline{X}$), an increase in valuation is denoted by an arrow, $\longrightarrow$, and justified by an inequality assigned a number. Numbers above an arrow apply to X. Numbers below the arrow apply to $\overline{X}$. As we see below, the ordering of the elements in $\overline{E}$ is the same as in E (also in $\overline{F}$ as in F). This explains the use of similar notation.

$$A. \quad \rho \xrightarrow{1} \overline{2\rho} \xrightarrow{2} (\sigma^2+1)\alpha \xrightarrow{1} \overline{2(\sigma^2+1)\alpha} \xrightarrow{1} 2\alpha \xrightarrow{1} \overline{4\alpha} \xrightarrow{6} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow{1} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow{4} 2\rho$$

In Case A, the valuation of ρ_m depends upon whether or not $b_2 = 4e_0 - b_1$. If $b_2 = 4e_0 - b_1$, $0 < b_1$ justifies 2 while $b_1 < 2e_0$ justifies 4. All other increases, including 2 and 4 for $b_2 \ne 4e_0 - b_1$, are justified by the inequalities listed below.

In Cases B through H, there is only one valuation of ρ_m.

$$B. \quad \rho \xrightarrow{1} \overline{2\rho} \xrightarrow{2} (\sigma^2+1)\alpha \xrightarrow{1} \overline{2(\sigma^2+1)\alpha} \xrightarrow{4} 2\alpha \xrightarrow{5} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow{6'} \overline{4\alpha} \xrightarrow{5} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow{4} 2\rho$$

$$C. \quad \rho \xrightarrow{1} \overline{2\rho} \xrightarrow{2} (\sigma^2+1)\alpha \xrightarrow{1} \overline{2(\sigma^2+1)\alpha} \xrightarrow{7} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow{5'} 2\alpha \xrightarrow{7} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow{5'} \overline{4\alpha} \xrightarrow{7} 2\rho$$

$$D. \quad \rho \xrightarrow{9} (\sigma^2+1)\alpha \xrightarrow{2'} \overline{2\rho} \xrightarrow{9} \overline{2(\sigma^2+1)\alpha} \xrightarrow{7} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow{5'} 2\alpha \xrightarrow{7} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow{5'} \overline{4\alpha} \xrightarrow{7} 2\rho$$

$$E = \overline{E}. \quad \rho \xrightarrow[11]{7'} \overline{2\alpha} \xrightarrow[8]{8} \overline{2\rho} \xrightarrow[13]{2} (\sigma^2+1)\alpha \xrightarrow[8]{8} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow[11]{7'} \overline{2(\sigma^2+1)\alpha} \xrightarrow[8]{8} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow[14]{7'} 2\alpha \xrightarrow[8]{8} 2\rho$$

$$F = \overline{F}. \quad \rho \xrightarrow[11]{7'} \overline{2\alpha} \xrightarrow[12]{10} (\sigma^2+1)\alpha \xrightarrow[13']{2'} \overline{2\rho} \xrightarrow[12]{10} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow[11]{7'} \overline{2(\sigma^2+1)\alpha} \xrightarrow[8]{8} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow[14]{7'} 2\alpha \xrightarrow[8]{8} 2\rho$$

$$G. \quad \rho \xrightarrow{9} (\sigma^2+1)\alpha \xrightarrow{12'} \overline{2\alpha} \xrightarrow{15} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow{12'} \overline{2\rho} \xrightarrow{9} \overline{2(\sigma^2+1)\alpha} \xrightarrow{8} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow{14} 2\alpha \xrightarrow{8} 2\rho$$

$$H. \quad \rho \xrightarrow{9} (\sigma^2+1)\alpha \xrightarrow{8} (\sigma+1)(\sigma^2+1)\alpha \xrightarrow{15'} \overline{2\alpha} \xrightarrow{8} \overline{2\rho} \xrightarrow{9} \overline{2(\sigma^2+1)\alpha} \xrightarrow{8} \overline{2(\sigma+1)(\sigma^2+1)\alpha} \xrightarrow{14} 2\alpha \xrightarrow{8} 2\rho$$

Numbered Inequalities: (1) $b_1 < 2e_0$, $b_2 < 4e_0$, $b_3 < 8e_0$. (2) $3b_2 > 4e_0 + 4b_1$. (2′) $3b_2 < 4e_0 + 4b_1$. (3) $4e_0 - 4b_1 < 3b_2$ (true for A–F since $b_2 \ge 2e_0 - b_1$). (4) $2b_2 < b_3$.

(5) $4e_0 - 2b_1 < b_2$. (5′) $4e_0 - 2b_1 > b_2$. (6) $4e_0 - 4b_1/3 < b_2$. (6′) $4e_0 - 4b_1/3 > b_2$. (7) $4e_0 - 4b_1 < b_2$. (7′) $4e_0 - 4b_1 > b_2$. (8) $b_1 > 0$. (9) $b_2 > 2b_1$. (10) $b_2 > 4e_0/3$ (true for A–F, since $b_2 \geq 3e_0/2$). (11) Since $b_2 > 2b_1$ and $b_3 \leq 8e_0 - 2b_1 - b_2$, $b_3 < 8e_0 - 4b_1$. (12) $8e_0 - 2b_2 < b_3$. (12′) $8e_0 - 2b_2 > b_3$. (13) $8e_0 + 4b_1 - 2b_2 < b_3$. (13′) $8e_0 + 4b_1 - 2b_2 > b_3$. (14) Since $b_2 > 2b_1$ and $b_3 \geq 3b_2 + 2b_1$, $b_3 > 2b_1 + 4b_2$. (15) $8e_0 - 4b_1 - 2b_2 < b_3$. (15′) $8e_0 - 4b_1 - 2b_2 > b_3$.

We leave it to the reader to verify Appendix B.

3.3.1. *Summary: Results of Basis Changes and Nakayama's Lemma.*
Basis Changes. Except in four rows,

$$C(2), D(2), E(2), F(2), \tag{3.2}$$

we find we may change the $\mathfrak{O}_T$-bases in Appendix B so that the Galois action upon each basis is ***as if*** ρ and $\overline{\rho}$ had been everywhere replaced by $(\sigma+1)\alpha$ and $\overline{(\sigma+1)\alpha}$. In the four exceptional cases there are nontrivial Galois relationships among the basis elements. This is explained in §3.3.5.

Nakayama's Lemma. We find, without loss of generality, that the set $\mathcal{S}$ of 'left–most' elements $\overline{X}$ (as in $\mathcal{S}'$ of §2.2.3) from each basis in Appendix B will serve as a $\mathfrak{O}_T[G]/\langle \mathrm{Tr}_{3,2}\rangle$-basis for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$, *except* that $\mathcal{S}$ contains both $\overline{(\sigma+1)(\sigma^2+1)\alpha}$ and $\overline{2\alpha}$ in $B(3)$, $C(3)$, $D(3)$.

At this point, the reader can skip the verification of these assertions, ignore Cases C through F, replace ρ with $(\sigma+1)\alpha$, and lift the Galois module structure off of the bases listed in Appendix B. See [CR90, §8] The result of the readers effort will be the statement of our main result in every case except those associated with (3.2).

3.3.2. *Trivial Difference.* The elements α_m, ρ_m (or ρ_m, $2\alpha_m$) from each basis in Appendix B provide a $\mathfrak{O}_T$-basis for $\mathfrak{P}_2^{\lceil i/2\rceil}/\mathfrak{P}_1^{\lceil i/4\rceil}$. We can change ρ_m by an element in $\mathfrak{P}_1^{\lceil i/4\rceil}$ and still have a $\mathfrak{O}_T$-basis. So when $\rho_m - (\sigma+1)\alpha_m \in \mathfrak{P}_1^{\lceil i/4\rceil}$, the difference between ρ_m and $(\sigma+1)\alpha_m$ is *trivial*.

Since $v_2((\sigma+1)\alpha) = v_2(\rho - (\sigma+1)\alpha)$, checking $\rho_m - (\sigma+1)\alpha_m \in \mathfrak{P}_1^{\lceil i/4\rceil}$ is equivalent to checking $v_3((\sigma+1)\alpha) \geq i$. In Case A, because $b_2 + b_1 \leq 4e_0$ we find that $v_3((1/2)\cdot(\sigma+1)(\sigma^2+1)\alpha_m) \leq v_3((\sigma+1)\alpha_m)$. Therefore, in $A(3)$ through $A(8)$, we may replace ρ_m by $(\sigma+1)\alpha_m$. We refrain from doing so in $A(8)$ as it may hamper our ability to determine the effect of $\mathrm{Tr}_{3,2}$ on $\overline{\rho}$. We will return to this issue in §3.3.4. In Case B, because $b_2 > 4e_0 - 2b_1$ we find $v_3(\overline{2\alpha}) < v_3((\sigma+1)\alpha)$. We may replace ρ in $B(3)$ through $B(8)$. For similar reasons, we refrain in $B(8)$. In Cases C and D, $b_3 > 2b_2 + 2b_1$ (since $b_3 = b_2 + 4e_0$ and $b_2 < 4e_0 - 2b_1$). As a consequence, $v_3(\overline{(\sigma+1)(\sigma^2+1)\alpha}) < v_3((\sigma+1)\alpha)$. We may replace ρ in $C(3)$ through $C(8)$, and in $D(3)$ through $D(6)$. In Cases E through H, we clearly have $v_3(\alpha) < v_3((\sigma+1)\alpha)$. We may replace ρ in $E(1)$ or $E(3)$ – $E(8)$, $F(1)$ or $F(3)$ – $F(8)$, $G(1)$ or $G(3)$ – $G(8)$, $H(1)$ or $H(3)$ – $H(8)$. We replace ρ everywhere that we may, *except* that we refrain for

$$A(8), B(8), C(8), D(7), D(8), E(8), F(7), F(8), G(6), G(7), H(6). \tag{3.3}$$

Now we consider the difference between $\overline{\rho}$ and $(\sigma+1)\overline{\alpha}$ and replace $\overline{\rho}$ with $(\sigma+1)\overline{\alpha}$ ($\overline{2\rho}$ with $(\sigma+1)\overline{2\alpha}$) in

$$E(1), F(1), G(1), G(8), H(1), H(7), H(8). \tag{3.4}$$

Since $(\sigma^4+1)\cdot[\overline{\rho}-(\sigma+1)\overline{\alpha}]=0$, we may use Lem 2.1(2) and find an element $\omega\in K_3$ with $v_3(\omega)=2b_2+b_1-2b_3$ so that $(\sigma^4-1)\omega=\overline{\rho}-(\sigma+1)\overline{\alpha}$. As long as $b_3<8e_0-3b_1$, which holds in Cases E through H, we have $v_3(\overline{\rho})=v_3(\overline{\rho}+2\omega)$. On the basis of valuation, we may replace $\overline{\rho}$ with $\overline{\rho}+2\omega$ and still have a basis (*i.e.* Observation (2)). Now since $(\overline{\rho}+2\omega)-(\sigma+1)\overline{\alpha}=(\sigma^4+1)\omega\in K_2$, we may replace $(\overline{\rho}+2\omega)$ with $(\sigma+1)\overline{\alpha}$ and still have a basis. All we need is $v_3((\sigma+1)\overline{\alpha})\geq i$. But this clearly holds since $v_3(\overline{\alpha})\geq i$.

3.3.3. *Nakayama's Lemma and an $\mathfrak{O}_T[G]/\langle\mathrm{Tr}_{3,2}\rangle$-basis for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$.* The collection of $\overline{X}$ in our bases provide an $\mathfrak{O}_T$-basis for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$. As in §2.2.3, whenever $\overline{X}$ and $(1/2)\cdot X$ appear in the same row, we may replace $\overline{X}$ with $\overline{X}-(1/2)\cdot X$ and still have a $\mathfrak{O}_T$-basis. Since $\mathrm{Tr}_{3,2}(\overline{X}-(1/2)\cdot X)=0$, we relabel and assume, without loss of generality, that for these $\overline{X}$'s, $\mathrm{Tr}_{3,2}\overline{X}=0$. Let $\mathcal{T}_{=0}$ denote this set (trace zero). Let $\mathcal{T}_{\neq 0}$ denote the set of $\overline{X}$'s with X in the same row. For each such $\overline{X}\in\mathcal{T}_{\neq 0}$, $\mathrm{Tr}_{3,2}\overline{X}\not\equiv 0 \bmod 2$. This is the set of trace *not* zero. Note that $\mathrm{Tr}_{3,2}\mathcal{T}_{\neq 0}$ is an $\mathfrak{O}_T/2\mathfrak{O}_T$-basis for $\mathrm{Tr}_{3,2}\mathfrak{P}_3^i/2\mathfrak{P}_2^{\lceil i/2\rceil}$. Following §2.2.3, we select from $\mathcal{T}_{\neq 0}$ a set $\mathcal{S}$ (notation as in §2.2.3) such that $\mathrm{Tr}_{3,2}\mathcal{S}$ is a $\mathfrak{O}_T/2\mathfrak{O}_T$-basis for $\mathrm{Tr}_{3,2}\mathfrak{P}_3^i/((\sigma-1)\mathrm{Tr}_{3,2}\mathfrak{P}_3^i+2\mathfrak{P}_2^{\lceil i/2\rceil})$. It turns out that just as in §2.2.3, $\mathcal{S}$ is the set of left-most $\overline{X}$ for which X appears in the same row, *except* that $\mathcal{S}$ contains both $\overline{X}$'s in $\mathcal{T}_{\neq 0}$ from $B(3)$, $C(3)$, $D(3)$.

Note that σ acts trivially (modulo 2) upon $(\sigma+1)(\sigma^2+1)\alpha$ and 2α in $B(3)$, $C(3)$ and $D(3)$. These elements are linearly independent over $\mathfrak{O}_T/2\mathfrak{O}_T[G]$. Since both contribute to the $\mathfrak{O}_T/2\mathfrak{O}_T$-basis for $\mathrm{Tr}_{3,2}\mathfrak{P}_3^i/2\mathfrak{P}_2^{\lceil i/2\rceil}$, both $\overline{(\sigma+1)(\sigma^2+1)\alpha}$ and $\overline{2\alpha}$ are in $\mathcal{S}$. When a row contributes exactly one $\overline{X}$ to $\mathcal{T}_{\neq 0}$, the phrase 'left–most' is unnecessary. Indeed σ acts trivially (modulo 2) on the lone $X=\mathrm{Tr}_{3,2}\overline{X}$, and since X is needed for the $\mathfrak{O}_T/2\mathfrak{O}_T$-basis for $\mathrm{Tr}_{3,2}\mathfrak{P}_3^i/2\mathfrak{P}_2^{\lceil i/2\rceil}$, $\overline{X}$ must appear in $\mathcal{S}$. Note this is the only situation to consider in Case A.

In the other cases, we need to show that each X, corresponding to the left–most $\overline{X}$ of $\mathcal{T}_{\neq 0}$, generates over $\mathfrak{O}_T/2\mathfrak{O}_T[G]$ all other elements in the same row (in $\mathrm{Tr}_{3,2}\mathcal{T}_{\neq 0}$). This is easy to see for rows $E(1)$, $E(5)$, $F(1)$, $F(5)$, $G(1)$, $G(5)$, $G(8)$ and $H(1)$, $H(5)$, $H(7)$, $H(8)$. More work is required for rows $D(7)$, $F(7)$, $G(6)$, $G(7)$, $H(6)$. Note that $\rho-(\sigma+1)\alpha_m=(\sigma^2+1)\alpha_{m-t}$ or $(\sigma+1)(\sigma^2+1)\alpha_{m-s}$ depending upon $b_2>3b_1$ or $b_2=3b_1$, respectively. If $\rho-(\sigma+1)\alpha_m=(\sigma+1)(\sigma^2+1)\alpha_{m-s}$, then $(\sigma-1)\rho=(\sigma^2+1)\alpha-2\alpha\equiv(\sigma^2+1)\alpha \bmod 2\mathfrak{P}_2^{\lceil i/2\rceil}$. So ρ generates $(\sigma^2+1)\alpha$. If $\rho-(\sigma+1)\alpha_m=(\sigma^2+1)\alpha_{m-t}$ the analysis is a little more involved. Note $(\sigma-1)\rho-(\sigma^2+1)\alpha\equiv(\sigma-1)(\sigma^2+1)\alpha_{m-t} \bmod 2\mathfrak{P}_2^{\lceil i/2\rceil}$. For m associated with $D(7)$, $F(7)$, $G(6)$, $G(7)$, $H(6)$, check that $m-t$ lies in $D(3)$, $F(4)$, $G(4)$, $H(4)$ or later. In any case $\overline{(\sigma+1)(\sigma^2+1)\alpha_{m-t}}\in\mathfrak{P}_3^i$. So $\overline{\rho}_m$ and another $\overline{X}$, namely $\overline{(\sigma+1)(\sigma^2+1)\alpha}_{m-t}$, combine together to generate $(\sigma^2+1)\alpha_m$.

Apply Lemma 2.2 and extend $\mathcal{S}$ to an $\mathfrak{O}_T[G]/\langle\mathrm{Tr}_{3,2}\rangle$-basis for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$. Except in Cases B, C, D (where a row contributes more than one element), we may assume that this basis is the set of left–most elements $\overline{X}$, one from each row.

3.3.4. *Essentially Trivial Difference.* In §3.3.2 we did not replace ρ by $(\sigma+1)\alpha$ in rows $A(1)$, $A(2)$, $B(1)$, $B(2)$, $C(1)$, $C(2)$, $D(1)$, $D(2)$, $E(2)$, $F(2)$, $G(2)$, $H(2)$. It was not clear that the difference $\rho-(\sigma+1)\alpha$ lay in $\mathfrak{P}_1^{\lceil i/4\rceil}$. Neither did we replace ρ by $(\sigma+1)\alpha$ in the rows listed in (3.3). In this section we remedy this situation.

We show, except in four cases, $C(2), D(2), E(2), F(2)$, we may change our basis so that the Galois action is ***as if*** ρ had been replaced by $(\sigma+1)\alpha$ ($\overline{\rho}$ by $(\sigma+1)\overline{\alpha}$).

We begin with *Case A*, explaining why the difference between ρ and $(\sigma+1)\alpha$ is *essentially trivial* and then determine the Galois module structure (to illustrate the process). Consider $A(1), A(2)$ and $A(8)$. Recall there are three expressions for ρ_m corresponding to $3b_1 < b_2 < 4e_0 - b_1$, $b_2 = 3b_1$, and $b_2 = 4e_0 - b_1$. Suppose $3b_1 < b_2 < 4e_0 - b_1$, and $\rho_m = (\sigma+1)\alpha_m + (\sigma^2-1)\alpha_{m-t}$. Consider ρ_m in $A(8)$. Since $b_1 + b_2 < 4e_0$, $v_3(\overline{\rho}_m) \leq v_3(\overline{2\alpha}_{m-t})$. So for m in $A(8)$, $m-t$ is in $A(4)$ or later. In any case, $(\sigma-1)\alpha_{m-t} = (\sigma+1)\alpha_{m-t} - 2\alpha_{m-t} \in \mathfrak{P}_3^i$ and $(1/2)(\sigma-1)(\sigma^2+1)\alpha_{m-t} = (1/2)(\sigma+1)(\sigma^2+1)\alpha_{m-t} - (\sigma^2+1)\alpha_{m-t} \in \mathfrak{P}_3^i$ (*i.e.* these elements are available). We replace α_m with $x = \alpha_m + (\sigma-1)\alpha_{m-t} - (1/2)(\sigma-1)(\sigma^2+1)\alpha_{m-t}$. Note $(\sigma+1)x = \rho$ and $(\sigma^2+1)x = (\sigma^2+1)\alpha_m$. The Galois action on x and ρ_m is the same as the Galois action on α_m and $(\sigma+1)\alpha_m$. It is ***as if*** ρ_m had been replaced by $(\sigma+1)\alpha_m$ and $\overline{\rho}_m$ by $(\sigma+1)\overline{\alpha}_m$. Now consider $A(1)$ and $A(2)$, $\rho_m = (\sigma+1)\alpha_m + (1/2)(\sigma^2-1)\alpha_{m-t+e_0}$. Since $v_2(\rho_m) < v_2((1/2)(\sigma+1)(\sigma^2+1)\alpha_{m-t+e_0})$, for m in $A(1)$ or $A(2)$, $m-t+e_0$ lies in $A(3)$ or later. In any case, $(1/2)(\sigma-1)(\sigma^2-1)\alpha_{m-t+e_0}$ is available. So in $A(1)$ and $A(2)$, we replace $2\alpha_m$ by $2\alpha_m - (1/2)(\sigma-1)(\sigma^2-1)\alpha_{m-t+e_0}$. The effect of this replacement on the Galois action is, again, the same ***as if*** we replaced ρ_m by $(\sigma+1)\alpha_m$.

Now suppose $b_2 = 3b_1$ and $\rho_m = (\sigma+1)\alpha_m + (\sigma+1)(\sigma^2+1)\alpha_{m-s}$. Note $s = b_1$. Starting with the smallest m such that $i \leq v_3(\overline{\rho}_m)$ we replace α_m by $\alpha_m + (1/2)(\sigma+1)\alpha_{m+e_0-b_1}$ so long as $m+e_0-b_1$ is associated with $A(8)$. If $i \leq v_3(\rho_{m-b_1})$, we replace α_m by $\alpha_m + (\sigma^2+1)\alpha_{m-b_1}$. In any case, we can systematically replace α_m by $x = \alpha_m + (1/2)(\sigma^2+1)\alpha_{m+e_0-b_1}$ or $\alpha_m + (\sigma^2+1)\alpha_{m-b_1}(1/2)(\sigma^2+1)\alpha_m$ by $(1/2)(\sigma^2+1)x$ and $(1/2)(\sigma+1)(\sigma^2+1)\alpha_m$ by $(1/2)(\sigma+1)(\sigma^2+1)x$. The Galois action after this change of basis is ***as if*** $\overline{\rho}_m = (\sigma+1)\overline{\alpha}_m$ and $\rho_m = (\sigma+1)\alpha_m$. Consider $A(1)$ and $A(2)$. Note $(\sigma-1)\rho_m = (\sigma-1)\cdot(\sigma+1)\alpha_m$. Moreover, for m associated with these two cases, $(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ and $(\sigma^2+1)\alpha_{m+e_0-b_1}$ are available elsewhere in our basis. So we replace $(\sigma^2+1)\alpha_m$ by $(\sigma^2+1)(\alpha_m+\alpha_{m+e_0-b_1})$ and $(\sigma+1)(\sigma^2+1)\alpha_m$ by $(\sigma+1)(\sigma^2+1)(\alpha_m+\alpha_{m+e_0-b_1})$. Note for m associated with $A(2)$, $m+e_0-b_1$ is associated with $A(3)$ or later. We achieve the desired effect by replacing $\overline{(\sigma+1)(\sigma^2+1)\alpha}_m$ with $\overline{(\sigma+1)(\sigma^2+1)\alpha}_m + \overline{(\sigma+1)(\sigma^2+1)\alpha}_{m+e_0-b_1}$.

This leaves $b_2 = 4e_0 - b_1$. Because this case is more complicated (recall Remark 3.5: ρ_m is 'torn' between α_m and $\alpha_{m+e_0-b_1}$), we first determine the Galois module structure for $b_2 < 4e_0 - b_1$. Each m in $A(1)$ results in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} (\mathcal{R}_3 \oplus \mathcal{H})$; m in $A(2)$ in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} \mathcal{H}_2$; m in $A(3)$ in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} (\mathcal{R}_3 \oplus \mathcal{M})$; m in $A(4)$ in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} \mathcal{M}_1$; m in $A(5)$ in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} (\mathcal{R}_3 \oplus \mathcal{L})$; m in $A(6)$ in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} \mathcal{L}_3$; m in $A(7)$ in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} (\mathcal{R}_3 \oplus \mathcal{I})$; m in $A(8)$ in an $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} \mathcal{I}_2$. Counting the number of m associated with each $A(j)$ yields the first column of Table 2.

Now consider $b_2 = 4e_0 - b_1$. Because $v_2(\rho_m) = 2b_2 - b_1 + 4m$, the number of m associated with $A(1)$ and $A(7)$ are different. The number for $A(7)$ is $e_0 - b_1$ too low, while $A(1)$ is $e_0 - b_1$ too high. We seem to be missing $e_0 - b_1$ of $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} \mathcal{I}$ and have $e_0 - b_1$ too many of $\mathfrak{O}_T \otimes_{\mathbb{Z}_2} \mathcal{H}$. Let us look at this more carefully. Note $\overline{\rho}_m$ in $A(8)$ maps (via $\mathrm{Tr}_{3,2}$) to

$$\rho_m = (\sigma+1)(\alpha_m - (1/2)(\sigma^2+1)\alpha_m) + \begin{cases} (1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1} \\ (\sigma+1)(\sigma^2+1)\alpha_{m-b_1} \end{cases}$$

So $\overline{\rho}_m$ maps into the $\mathfrak{O}_T$-module spanned by $\alpha_m-(1/2)(\sigma^2+1)\alpha_m$ and $(\sigma+1)(\alpha_m-(1/2)(\sigma^2+1)\alpha_m)$ along with *either* $(1/2)(\sigma^2+1)\alpha_{m+e_0-b_1}$ and $(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ *or* $(\sigma^2+1)\alpha_{m-b_1}$ and $(\sigma+1)(\sigma^2+1)\alpha_{m-b_1}$. In any case, the elements $(1/2)(\sigma^2+1)\alpha_m$ and $(1/2)(\sigma+1)(\sigma^2+1)\alpha_m$ for $\lceil(i+b_3-4b_2+2b_1)/8\rceil \leq m \leq \lceil(i+b_3-4b_2+2b_1)/8\rceil+e_0-b_1-1$ are not associated with a $\overline{\rho}_m$ in $A(8)$. The ρ_m in $A(1)$ map to $(\sigma^2-1)\alpha_m$ (under $(\sigma-1)$) and so $(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ (under (σ^2+1)) yielding a $\mathcal{H}$, unless $m+e_0-b_1$ is associated with $A(2)$. In fact, there are e_0-b_1 ρ_m that map into $A(2)$ under (σ^2+1). For each m in $A(2)$ we have $(\sigma^4+1)\overline{(\sigma+1)(\sigma^2+1)\alpha_m} = (\sigma^2+1)\rho_{m-e_0+b_1} = (\sigma+1)(\sigma^2+1)\alpha_m$, yielding a copy of $\mathcal{H}_2$. But for the last e_0-b_1 elements ρ_m in $A(2)$, namely those m such that $m+e_0-b_1$ is in $A(3)$ we may replace ρ_m by $\rho_m-(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$. For each of these m we have the $\mathfrak{O}_T[G]$-submodule spanned by $\rho_m-(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ and $(\sigma^2-1)\alpha_m$. These e_0-b_1 together with the elements left out of a module in $A(8)$ yield a e_0-b_1 copies of $\mathcal{I}$, precisely making up the counts.

Cases $B-H$: In the remaining cases, we only have two situations: $b_2=3b_1$ and $3b_1<b_2<4e_0-b_1$. Consider $3b_1<b_2<4e_0-b_1$ first, and $\rho_m=(\sigma+1)\alpha_m+(\sigma^2\pm1)\alpha_{m-t}$ where we may choose between $\pm$ as we like. We are concerned with the image of the trace, $\mathrm{Tr}_{3,2}$, in particular $\mathrm{Tr}_{3,2}\overline{\rho}_m=(\sigma+1)\alpha_m+(\sigma^2+1)\alpha_{m-t}$, for $\overline{\rho}_m$ appearing in $B(8)$, $C(8)$, $D(7)$, $D(8)$, $E(8)$, $F(7)$, $F(8)$, $G(6)$, $G(7)$, and $H(6)$. Note if $\overline{(\sigma^2+1)\alpha_{m-t}}\in\mathfrak{P}_3^i$, we may replace $\overline{\rho}_m$ with $\overline{\rho}_m-\overline{(\sigma^2+1)\alpha_{m-t}}$. So if $(\sigma^2+1)\alpha_{m-t}$ appears in $B(6)$, $C(6)$, $D(6)$, $E(5)$, $F(5)$, $G(5)$, $H(5)$ or later we may replace ρ_m with $(\sigma+1)\alpha_m$ and $\overline{\rho}_m$ with $\overline{\rho}_m-\overline{(\sigma^2+1)\alpha_{m-t}}$. The later replacement exhibits the same Galois action as a replacement of $\overline{\rho_m}$ by $(\sigma+1)\overline{\alpha}_m$. Without loss of generality we will call it a replacement of $\overline{\rho_m}$ by $(\sigma+1)\overline{\alpha}_m$.

Since $b_2\leq 4e_0-b_1$, $v_3(\overline{2\alpha_m})\geq v_3(\overline{\rho}_m)$. What happens when $(\sigma^2+1)\alpha_{m-t}$ appears in $B(3)-B(5)$, $C(3)-C(5)$, $D(3)-D(5)$, $E(4)$, $F(6)$, $G(4)$, $H(6)$? In this case $(\sigma-1)\alpha_{m-t}=(\sigma+1)\alpha_{m-t}-2\alpha_{m-t}\in\mathfrak{P}_3^i$. In $B(8)$, $C(8)$, $D(7)$, $D(8)$, $E(8)$, $F(8)$, $G(6)$, $G(7)$, $H(6)$, we replace α_m with $\alpha_m+(\sigma-1)\alpha_{m-t}$, and $(\sigma^2+1)\alpha_m$ with $(\sigma^2+1)\alpha_m+(\sigma-1)(\sigma^2+1)\alpha_{m-t}$. Note $\rho_m=(\sigma+1)\cdot[\alpha_m+(\sigma-1)\alpha_{m-t}]$. The Galois action upon these basis elements: $\mathrm{Tr}_{3,2}\overline{\rho}_m=\rho_m=(\sigma+1)\cdot[\alpha_m+(\sigma-1)\alpha_{m-t}]$, $(\sigma^2+1)\cdot[\alpha_m+(\sigma-1)\alpha_{m-t}]=(\sigma^2+1)\alpha_m+(\sigma-1)(\sigma^2+1)\alpha_{m-t}$, and $(\sigma+1)\cdot[(\sigma^2+1)\alpha_m+(\sigma-1)(\sigma^2+1)\alpha_{m-t}]=(\sigma^2+1)\rho_m=(\sigma+1)(\sigma^2+1)\alpha_m$, is similar to the Galois action upon: $(\sigma+1)\overline{\alpha}_m$, α_m, $(\sigma+1)\alpha_m$, $(\sigma^2+1)\alpha_m$, $(\sigma+1)(\sigma^2+1)\alpha_m$. We may assume $(\sigma+1)\overline{\alpha}_m$ and $(\sigma+1)\alpha_m$ appear instead of $\overline{\rho}_m$ and ρ_m.

Now consider the appearance of ρ in $B(1)$, $B(2)$, $C(1)$, $D(1)$, $G(2)$, $H(2)$. Suppose $\rho_m=(\sigma+1)\alpha_m+(1/2)\cdot(\sigma^2-1)\alpha_{m+e_0-t}$. One may check $v_3(\rho_m)\leq v_3((1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-t}$ and $v_3(\overline{2\rho_{m+e_0-t}})\leq v_3(\overline{4\alpha_m})$. So $(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-t}$ appears in $B(4)-B(7)$, $C(6)-C(8)$ or $D(6)-D(8)$. Note in these sets of elements, ρ_{m+e_0-t} has already been replaced by $(\sigma+1)\alpha_{m+e_0-t}$. Importantly, $(1/2)(\sigma-1)(\sigma^2+1)\alpha_{m+e_0-t}$ along with $(\sigma-1)\alpha_{m+e_0-t}$ are available to us. We replace $2\alpha_m$ with $2\alpha_m-(1/2)(\sigma-1)(\sigma^2+1)\alpha_{m+e_0-t}+(\sigma-1)\alpha_{m+e_0-t}=2\alpha_m-(1/2)(\sigma-1)(\sigma^2-1)\alpha_{m+e_0-t}$ in $B(1)$, $B(2)$, $C(1)$ and $D(1)$. The effect of this change of basis is the same as if we replaced ρ_m by $(\sigma+1)\alpha_m$.

Now consider $G(2)$ and $H(2)$. Again $\rho_m=(\sigma+1)\alpha_m+(1/2)(\sigma^2-1)\alpha_{m+e_0-t}$. In G and H, $b_3\leq 8e_0-2b_2$. As a result, $v_3(\rho_m)\leq v_3(\overline{(\sigma-1)\alpha_{m+e_0-t}})$. Note we refer to $\overline{(\sigma-1)\alpha_{m+e_0-t}}$ and not $(\sigma-1)\overline{\alpha_{m+e_0-t}}$. The valuation of the first is b_1 more than the valuation of the second. As one may check $v_3(\rho_m)\leq v_3((1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-t})$, so $(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-t}$ appears in $G(7)$, $G(8)$ or $H(8)$. If

$(\sigma+1)(\sigma^2+1)\alpha$ appeared in $G(1)$ or $H(1)$, $(\sigma^2+1)\alpha_{m-t}$ would be available and so we would replace ρ_m with $\rho_m - \overline{(\sigma^2+1)\alpha_{m-t}}$. If $(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-t}$ appears in $G(7)$, then we may assume $\overline{(\sigma-1)\alpha_{m+e_0-t}}$ appears there instead of $\overline{\rho}_{m+e_0-t}$, because $v_3(\overline{(\sigma^2+1)\alpha_{m+e_0-2t}}) = v_3(\overline{(\sigma-1)\alpha_{m+e_0-2t}}) \geq i$, and we would have replaced $\overline{\rho}_{m+e_0-t}$ previously in our discussion with $\overline{\rho}_{m+e_0-t} - \overline{(\sigma^2+1)\alpha_{m+e_0-2t}}$. We may now replace $\overline{2\alpha}_m$ with $\overline{2\alpha}_m - \overline{(\sigma-1)\alpha_{m+e_0-t}}$. We replace $(\sigma^2+1)\alpha_m$ with $(\sigma^2+1)\alpha_m + (1/2)(\sigma-1)(\sigma^2)\alpha_{m+e_0-t}$. We may assume without loss of generality that $(\sigma+1)\alpha_m$ appears in $G(2)$ and $H(2)$ instead of ρ_m.

Now we work with Cases B through H under the assumption $b_2 = 3b_1$. So $\rho_m = (\sigma+1)\cdot[\alpha_m + (\sigma^2+1)\alpha_{m-b_1}]$. First note if $(\sigma+1)(\sigma^2+1)\alpha_{m-b_1}$ appears in $B(2)$, $C(3)$, $D(3)$, $E(4)$, $F(4)$, $G(4)$, $H(4)$, or later we may replace $\overline{\rho}_m$ in $B(8)$, $C(8)$, $D(7)$, $D(8)$, $E(7)$, $F(7)$, $G(5)$, $G(6)$, $H(5)$ with $\overline{\rho}_m - \overline{(\sigma+1)(\sigma^2+1)\alpha}_m$. Suppose $(\sigma^2+1)\alpha_{m-b_1}$ appears elsewhere. In B, these elements can appear in $B(1)$, $B(2)$, or as $(1/2)\cdot(\sigma^2+1)\alpha_{m+e_0-b_1}$ elsewhere in $B(8)$. In cases C through H, since $b_1 < 4e_0/5$, $v_3(\overline{\rho}_m) \leq v_3(\rho_{m-b_1})$. So $(\sigma^2+1)\alpha_{m-b_1}$ appears in $C(1)$, $C(2)$, $D(1)$, $D(2)$, $E(2)$, $E(3)$, $F(2)$, $F(3)$, $G(2)$, $G(3)$, $H(2)$, $H(3)$. In these cases, we may either replace α_m with $\alpha_m + (1/2)\cdot(\sigma^2+1)\alpha_{m+e_0-b_1}$ or $\alpha_m + (\sigma^2+1)\alpha_{m+-b_1}$. If for example, we replace α_m with $\alpha_m + (\sigma^2+1)\alpha_{m-b_1}$, $(\sigma^2+1)\alpha_m$ with $(\sigma^2+1)\alpha_m + 2(\sigma^2+1)\alpha_{m-b_1}$, and $(\sigma+1)(\sigma^2+1)\alpha_m$ with $(\sigma+1)(\sigma^2+1)\alpha_m + 2(\sigma+1)(\sigma^2+1)\alpha_{m-b_1}$, then the Galois action on this new basis is the same *as if* $(\sigma+1)\overline{\alpha}_m$ and $(\sigma+1)\alpha_m$ appear instead of $\overline{\rho}_m$ and ρ_m.

We now concern ourselves with $B(1)$, $B(2)$, $C(1)$, $D(1)$, $G(2)$ and $H(2)$. Check $v_3((\sigma^2+1)\alpha_{m+e_0-b_1}) \geq v_3(\rho_m)$. We replace $(\sigma^2+1)\alpha_m$ with $(\sigma^2+1)\alpha_m + (\sigma^2+1)\alpha_{m+e_0-b_1}$, and $(\sigma+1)(\sigma^2+1)\alpha_m$ with $(\sigma+1)(\sigma^2+1)\alpha_m + (\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$. In $B(2)$, $v_3(\overline{(\sigma+1)(\sigma^2+1)\alpha}_{m+e_0-b_1}) \geq v_3(\overline{(\sigma+1)(\sigma^2+1)\alpha}_m)$, we replace $(\sigma+1)(\sigma^2+1)\alpha_m$ with $(\sigma+1)(\sigma^2+1)\alpha_m + (\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$. All this has the same effect upon the Galois action as a replacement of ρ_m by $(\sigma+1)\alpha_m$.

3.3.5. *Non-Trivial Difference.* We consider ρ in $C(2)$, $D(2)$, $E(2)$, $F(2)$.

First consider the case $b_2 = 3b_1$ where $\rho_m = (\sigma+1)\alpha_m + (1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$. Note C and E do not intersect the line $b_2 = 3b_1$. We focus on $D(2)$, $F(2)$. In D with $b_2 = 3b_1$, we have $b_1 < 4e_0/5$. So $v_3(\overline{2\alpha}_m) \leq v_3(\alpha_{m+e_0-b_1})$. Since $v_3(\overline{2(\sigma+1)(\sigma^2+1)\alpha}_m) \leq v_3((\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1})$, for m associated with $D(2)$, $(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ appears in $D(4)$, or $(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ appears in $D(5)$ or later. If $(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ is available, we may replace ρ_m with $(\sigma+1)\alpha_m$. The Galois action when m is in $D(2)$ and $m+e_0-b_1$ is in $D(4)$ is our primary concern. But first consider F (or $\overline{F}$) with $b_2 = 3b_1$. Note then $b_3 \leq 8e_0 + 2b_2 - 8b_1$. So $v_3(\rho_m) \leq v_3(\overline{(\sigma+1)(\sigma^2+1)\alpha}_{m+e_0-b_1})$. Since $b_3 \leq 8e_0 + 2b_2 - 8b_1$, $v_3(\alpha_m) \leq v_3(\overline{2(\sigma^2+1)\alpha}_{m+e_0-b_1})$. So for m associated with $F(2)$, $(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$ appears in $F(4)$, or in $F(5)$ or later. If $m+e_0-b_1$ is associated with $F(5)$ or later, we have $\overline{(\sigma^2+1)\alpha}_{m+e_0-b_1}$ available. We replace $\overline{2\alpha}_m$ with $\overline{2\alpha}_m + \overline{(\sigma^2+1)\alpha}_{m+e_0-b_1}$. We replace $(\sigma^2+1)\alpha_m$ and $(\sigma+1)(\sigma^2+1)\alpha_m$ with $(\sigma^2+1)\alpha_m + (\sigma^2+1)\alpha_{m+e_0-b_1}$ and $(\sigma+1)(\sigma^2+1)\alpha_m + (\sigma+1)(\sigma^2+1)\alpha_{m+e_0-b_1}$. The effect of these changes upon the Galois action is the same as the replacement of ρ_m by $(\sigma+1)\alpha_m$. This leaves the situation when m belongs to $D(2)$, $F(2)$ while $m+e_0-b_1$ belongs to $D(4)$, $F(4)$. In both of these cases, we replace $\overline{(\sigma+1)(\sigma^2+1)\alpha}_{m+e_0-b_1}$ with $\overline{(\sigma+1)(\sigma^2+1)\alpha}_{m+e_0-b_1} + (\sigma+1)\overline{2\alpha}_m - \rho_m$. This

new basis element has trace, $\mathrm{Tr}_{3,2}$, zero. For each such pair $(m, m+e_0-t)$ we get a copy of $\mathcal{H}_1\mathcal{G} \oplus \mathcal{R}_3$.

Let us now turn to the case where $3b_1 < b_2 < 4e_0 - b_1$ and $\rho_m = (\sigma+1)\alpha_m + (1/2)(\sigma^2+1)\alpha_{m+e_0-t}$. Consider cases C and E. Because $v_3(\overline{2\alpha}) \le v_3(\overline{(\sigma^2+1)\alpha})$, if m appears in $C(2)$, then $m+e_0-t$ appears in $C(6)$ or later. Since $v_3((\sigma^2+1)\alpha_{m+e_0-t} > v_3(\overline{2(\sigma+1)(\sigma^2+1)\alpha})$, not every $m+e_0-t$ is in $C(6)$ when m is in $C(2)$. Since $v_3(\rho) \le v_3((1/2)(\sigma^2+1)\alpha)$, if m appears in $E(2)$, then $m+e_0-t$ appears in $E(6)$ or later. Since $v_3((\sigma^2+1)\alpha_{m+e_0-t} > v_3(2\alpha)$, some $m+e_0-t$ spill over into $C(7)$. Consequently, whenever a pair $(m, m+e_0-t)$ has m in $C(2)$, $E(2)$ while $m+e_0-t$ is in $C(6)$, $E(6)$ we get a copy of $\mathcal{H}_1\mathcal{L} \oplus \mathcal{R}_3$.

Consider cases D and F (including $\overline{F}$). Consider D first. Since $v_3(\overline{2\alpha_m}) < v_3(\overline{(\sigma^2+1)\alpha}_{m+e_0-t})$, for m in $D(2)$, $m+e_0-t$ lands in $D(6)$ or later. Note since $v_3(\overline{2\alpha_m}) > v_3(\rho_{m+e_0-t})$, some $m+e_0-t$ land in $D(6)$. Since $v_3((\sigma^2+1)\alpha_{m+e_0-t}) > v_3(\overline{2(\sigma^2+1)\alpha}_m)$, the collection of $m+e_0-t$ overlap into $D(8)$. When $m+e_0-t$ is in $D(8)$, the element $(1/2)(\sigma^2+1)\alpha_{m+e_0-t}$ is available and we replace ρ_m by $\rho_m - (1/2)(\sigma^2+1)\alpha_{m+e_0-t} = (\sigma+1)\alpha_m$. For each pair $(m, m+e_0-t)$ such that m is associated with $D(2)$ and $m+e_0-t$ is associated with $D(6)$, we get a copy of $\mathcal{H}_1\mathcal{L} \oplus \mathcal{R}_3$. What we are principally concerned with is what happens when for m in $D(2)$, $m+e_0-t$ is in $D(7)$. In this case, because $\rho_{m+e_0-t} = (\sigma+1)\alpha_{m+e_0-t} + (\sigma^2+1)\alpha_{m+e_0-2t}$, there is some new interaction to consider.

Suppose m is in $D(2)$, while $m+e_0-t$ is in $D(7)$. Since $v_3(\overline{\rho}_{m+e_0-t}) \le v_3(\alpha_{m+e_0-2t})$ and $v_3(\overline{2(\sigma+1)(\sigma^2+1)\alpha}_m) \le v_3((\sigma+1)(\sigma^2+1)\alpha_{m+e_0-2t})$, for m in $D(2)$ and $m+e_0-t$ in $D(7)$, we find $m+e_0-2t$ is associated with $D(4)$, or $D(5)$ or later. Consider m in $D(2)$, $m+e_0-t$ in $D(7)$, and $m+e_0-2t$ in $D(4)$. Perform change of basis: Replace $\overline{2\alpha}_m$ with $\overline{2\alpha}_m + \overline{2\alpha}_{m+e_0-t} - \overline{2\alpha}_{m+e_0-t}$, ρ_m with $\rho_m - \alpha_{m+e_0-t}$, $(\sigma^2+1)\alpha_m$ with $(\sigma^2+1)\alpha_m + (\sigma^2+1)\alpha_{m+e_0-2t} + 1/2(\sigma-1)(\sigma^2+1)\alpha_{m+e_0-t}$, and $(\sigma+1)(\sigma^2+1)\alpha_m$ with $(\sigma+1)(\sigma^2+1)\alpha_m + (\sigma+1)(\sigma^2+1)\alpha_{m+e_0-2t}$. The effect of these base changes upon the Galois action is the same as if we were to replace ρ_m with $(\sigma+1)\alpha_m - (1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-2t}$. Notice the similarity between this expression and the expression for ρ_m used when $b_2 = 3b_1$. Consequently, this scenario results in copies of $\mathcal{H}_1\mathcal{G} \oplus \mathcal{R}_3$. (Note if $b_2 = 3b_1$, then $2t = b_1$.)

In the alternative situation, when m is in $D(2)$, $m+e_0-t$ in $D(7)$, and $m+e_0-2t$ is in $D(5)$ or later, we perform the same basis changes. Except, since the element $(1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-2t}$ is available, we replace ρ_m with $\rho_m - \alpha_{m+e_0-t} + (1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-2t}$. The effect of this alternative basis change upon the Galois action is the same as a simple replacement of ρ_m with $(\sigma+1)\alpha_m$. We now turn our attention to Cases F and $\overline{F}$. Since $0 < 2b_1$, $v_3(\rho_m) < v_3((1/2)(\sigma+1)(\sigma^2+1)\alpha_{m+e_0-t}$. So for m associated with $F(2)$, $m+e_0-t$ is associated with $F(6)$ or later. We leave it to the reader to check that $m+e_0-t$ lands in $F(6)$ or $F(7)$. If $m+e_0-t$ is associated with $F(7)$, then $m+e_0-2t$ lands in $F(4)$ or $F(5)$. In any case, all this is analogous to D.

3.4. **The Galois module structure under *stable* ramification.** For $p=2$, *stable ramification* $b_1 \ge e_0$ is nearly *strong ramification* $b_1 > (1/2)\cdot pe_0/(p-1)$, (the conditions differ only when e_0 is odd – K_0 tame over $\mathbb{Q}_2$). In [Eld02], the structure of the ring of integers was determined under *strong ramification* for any prime p. We revisit that argument extending it to ambiguous ideals and the case $b_1 = e_0$.

Following §2.1, $\mathfrak{P}_2^{\lceil i/2\rceil}/\mathfrak{P}_1^{\lceil i/4\rceil} \cong \big(\mathfrak{O}_T[\sigma]/\langle\sigma^2+1\rangle\big)^{e_0}$. So e_0 elements generate $\mathfrak{P}_2^{\lceil i/2\rceil}/\mathfrak{P}_1^{\lceil i/4\rceil}$ over $\mathfrak{O}_T[G]$. Use Lemmas 3.6, 3.7 to select elements, α, with odd valuation a such that $\lceil i/2\rceil \le a \le \lceil i/2\rceil + 2e_0 - 1$. Each of these e_0 elements gives rise (via the action of $(\sigma\pm1)$) to another element, ρ in K_2, with odd valuation, $a+(b_2-b_1) = a+2e_0$. These α along with their Galois translates, $\rho \equiv (\sigma\pm 1)\alpha \bmod \mathfrak{P}_1^{\lceil i/4\rceil}$, have valuations in one–to–one correspondence (via v_2) with the odd integers in $\lceil i/2\rceil,\dots,4e_0+\lceil i/2\rceil-1$, and as a result serve as a $\mathfrak{O}_T$-basis for $\mathfrak{P}_2^{\lceil i/2\rceil}/\mathfrak{P}_1^{\lceil i/4\rceil}$. The α provide a $\mathfrak{O}_T[G]/\langle\mathrm{Tr}_{2,1}\rangle$-basis.

We need this basis for $\mathfrak{P}_2^{\lceil i/2\rceil}/\mathfrak{P}_1^{\lceil i/4\rceil}$ to be compatible with our $\mathfrak{O}_T$-basis for $\mathfrak{P}_1^{\lceil i/4\rceil}$ (as determined as in §2.2.1), as well as our $\mathfrak{O}_T[G]/\langle\mathrm{Tr}_{3,2}\rangle$-basis for $\mathfrak{P}_3^{i}/\mathfrak{P}_2^{\lceil i/2\rceil}$. First we consider compatibility with $\mathfrak{P}_1^{\lceil i/4\rceil}$. The $\mathfrak{O}_T$-basis for $\mathfrak{P}_1^{\lceil i/4\rceil}$ consists of pairs: either $((\sigma+1)\eta,\eta)$ or $((\sigma+1)\eta,2\eta)\in K_0\times K_1$ where $v_1(\eta)$ is odd. Because of Lemma 2.1 each coordinate uniquely determines the other. Now consider pairs where the valuation v_3 of both elements is bound between i and $8e_0+i-1$. For example, pairs of the form $((\sigma+1)\eta,\eta)$ appear for $\lceil i/4\rceil \le v_1(\eta)\le 2e_0+\lceil i/4\rceil-b_1-1$, while pairs of the form $((\sigma+1)\eta,2\eta)$ appear for $\lceil i/4\rceil-b_1\le v_1(\eta)\le\lceil i/4\rceil-1$. The coordinates of all pairs provides us with an $\mathfrak{O}_T$ basis for $\mathfrak{P}_1^{\lceil i/4\rceil}$. Each α with $v_2((\sigma^2+1)\alpha)\le 4e_0+\lceil i/2\rceil-1$ determines (via $(\sigma^2+1)\alpha\in K_1$) a pair of elements in the $\mathfrak{O}_T$-basis for $\mathfrak{P}_1^{\lceil i/4\rceil}$. If $v_1((\sigma^2+1)\alpha)$ is odd, then α determines a pair of the form $((\sigma+1)\eta,2\eta)$. If even, it determines a pair of the form $((\sigma+1)\eta,\eta)$. In general for α with $v_2((\sigma^2+1)\alpha)\ge 4e_0+\lceil i/2\rceil$, $v_2(1/2(\sigma^2+1)\alpha)\ge\lceil i/2\rceil$. So $1/2(\sigma^2+1)\alpha$ is available and we may replace α in by $\alpha-1/2(\sigma^2+1)\alpha$ and still have a basis. Note $(\sigma^2+1)\big(\alpha-1/2(\sigma^2+1)\alpha\big)=0$. So we can assume, without loss of generality, $(\sigma^2+1)\alpha=0$. This posses no complication, unless $(\sigma\pm1)\alpha=\mu+\rho$ with ρ in the image of $\mathrm{Tr}_{3,2}\mathfrak{P}_3^i$. In other words, $v_2(\rho)\ge\lfloor(b_3+i+1)/2\rfloor$. (Note for α with $v_2((\sigma^2+1)\alpha)\le 4e_0+\lceil i/2\rceil-1$ and $(\sigma\pm1)\alpha=\mu+\rho$, we have $v_2(\rho)<\lfloor(b_3+i+1)/2\rfloor$.) For these α (actually $\alpha-1/2(\sigma^2+1)\alpha$), μ (actually $\mu-(\sigma\pm1)1/2(\sigma^2+1)\alpha$) will determine a pair $((\sigma+1)\eta,2\eta)$ or $((\sigma+1)\eta,\eta)$ in our $\mathfrak{O}_T$-basis for $\mathfrak{P}_1^{\lceil i/4\rceil}$. We need simply to show μ and $\mu-(\sigma\pm1)1/2(\sigma^2+1)\alpha$ have the same properties. We leave it to the reader to do this (use Lemma 3.6 and 3.7 to show that the valuations are the same, that $\mu-(\sigma\pm1)1/2(\sigma^2+1)\alpha\in K_0$ if and only if $\mu\in K_0$). The only issue that remains is whether there can be any conflict between a pair of basis elements for $\mathfrak{P}_1^{\lceil i/4\rceil}$ determined directly, via $(\sigma^2+1)\alpha$, and a pair determined indirectly via $\mu=(\sigma\pm1)\alpha-\rho$. Note any element in the image of the trace, $\mathrm{Tr}_{2,1}$, has valuation that is larger than the valuation of every $\mu\in K_1$ that arises from the expression for a Galois translate $\rho=(\sigma\pm1)\alpha-\mu$.

We select our $\mathfrak{O}_T[G]$-basis for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$ now. There is one element X in our $\mathfrak{O}_T$-basis for $\mathfrak{P}_2^{\lceil i/2\rceil}$ for each valuation v_2 in

$$\lfloor(i+b_3+1)/2\rfloor,\dots,4e_0+\lceil i/2\rceil-1. \tag{3.5}$$

The reader may check for $v_2(X)$ even, $X=(\sigma^2+1)\alpha$ for some α in our $\mathfrak{O}_T[G]$-basis for $\mathfrak{P}_2^{\lceil i/2\rceil}/\mathfrak{P}_1^{\lceil i/4\rceil}$. For $v_2(X)$ odd, since $\lceil i/2\rceil+(b_2-b_1)<\lfloor(i+b_3+1)/2\rfloor$, $X=\rho=(\sigma\pm1)\alpha-\mu$ also for some α. Use Lem 2.1 to create elements $\overline{X}\in\mathfrak{P}_3^i$ such that $\mathrm{Tr}_{3,2}\overline{X}=X$ and $v_3(\overline{X})=v_3(X)-b_3$. Note the elements $(\sigma^2+1)\alpha$ and μ (from each case) have expressions in terms of our $\mathfrak{O}_T$-basis for $\mathfrak{P}_1^{\lceil i/4\rceil}$. These expressions depend solely upon the valuations of $(\sigma^2+1)\alpha$ and μ.

Before we move on to our result, we should say something about our basis for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$. Since $\mathfrak{O}_T[\sigma]/\langle\sigma^4+1\rangle$ is a principal ideal domain, $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$ is free over $\mathfrak{O}_T[\sigma]/\langle\sigma^4+1\rangle$ of rank e_0. Given elements of K_2 with valuation v_2 listed in (3.5) we may use Lem 2.1(2) to find elements, $\rho\in\mathfrak{P}_3^i$, whose images under the trace, $\mathrm{Tr}_{3,2}$, lie one–to–one correspondence (via valuation) with (3.5). Refer to this set of elements in $\mathfrak{P}_3^i$ as $\mathcal{S}$. One can check $b_1+\lfloor(i+b_3+1)/2\rfloor>4e_0+\lceil i/2\rceil$. Therefore $(\sigma-1)\mathrm{Tr}_{3,2}\mathfrak{P}_3^i\subseteq 2\mathfrak{P}_2^{\lceil i/2\rceil}$. Since $\mathrm{Tr}_{3,2}\mathcal{S}$ is an $\mathfrak{O}_T$-basis for $\mathrm{Tr}_{3,2}\mathfrak{P}_3^i\subseteq 2\mathfrak{P}_2^{\lceil i/2\rceil}$ and σ acts trivially upon $\mathrm{Tr}_{3,2}\mathfrak{P}_3^i\subseteq 2\mathfrak{P}_2^{\lceil i/2\rceil}$ we may use Lemma 2.2 and extend $\mathcal{S}$ to an $\mathfrak{O}_T[G]/\langle\sigma^4+1\rangle$-basis for $\mathfrak{P}_3^i/\mathfrak{P}_2^{\lceil i/2\rceil}$.

At this point we may put the preceding discussion together with our work in §2.2.3 (that determines the structure of $\mathfrak{P}_2^{\lceil i/2\rceil}$) and determine the Galois module structure of $\mathfrak{P}_3^i$. We need to express the image of $\mathcal{S}$ under the trace, $\mathrm{Tr}_{3,2}$, in terms of our $\mathfrak{O}_T[G]$-basis for $\mathfrak{P}_2^{\lceil i/2\rceil}$. This is the same as a determination of the expression (in terms of Galois generators of $\mathfrak{P}_2^{\lceil i/2q\rceil}$) for each valuation in (3.5). First note under stable ramification, $b_2>4e_0-2b_1$ so the structure of $\mathfrak{P}_2^{\lceil i/2\rceil}$ is determined by the basis listed as Case B in §2.2.3. However it is more convenient for us to use the basis listed as Case A in Appendix B. To translate between the two bases, note in the elements $\overline{\alpha}$, $\overline{(\sigma+1)\alpha}$, α, $(\sigma+1)\alpha$ from §2.2.3 are referred to as α, ρ, $(\sigma^2+1)\alpha$, $(\sigma+1)(\sigma^2+1)\alpha$ in §3.1 and then in Appendix B. So row $B(1)$ in §2.2.3 corresponds with a pair of rows $A(7)$ and $A(8)$ in Appendix B. Moreover $B(2)$ corresponds to rows $A(1)$ and $A(2)$, $B(3)$ corresponds to $A(3)$ and $A(4)$, and $B(4)$ corresponds to $A(5)$ and $A(6)$.

There are four types of expression with valuation listed in (3.5). If the valuation a satisfies $a-(b_2-2b_1)\equiv 0 \bmod 4$ then a is the valuation of a Galois translate ρ where the difference between $(\sigma\pm1)\alpha$ and ρ is an element $(\sigma+1)\mu\in K_0$ where μ is in the basis for $\mathfrak{P}_1^{i/4}$. Note each such a corresponds with the appearance of $\mathcal{I}_2$ in the $\mathfrak{O}_T[G]$ decomposition of $\mathfrak{P}_3^i$. Counting such a one finds the same count as in $A(8)$. Note therefore $A(7)$ counts the number of $\mathcal{I}$ that are not mapped to under the trace, $\mathrm{Tr}_{3,2}$, from $\mathfrak{P}_3^i$.

Each valuation a satisfying $a\equiv 0 \bmod 4$ is the valuation of $(\sigma^2+1)\alpha=(\sigma+1)\mu$ for some α in the basis for $\mathfrak{P}_2^{\lceil i/2\rceil}$ and μ in the basis for $\mathfrak{P}_1^{i/4}\mathfrak{P}_0^{i/8}$. So each such a, corresponds with the appearance of an $\mathcal{H}_2$. A count of such a equals the count in $A(2)$. Note $A(1)$ counts the number of $\mathcal{H}$ not interacted with. Each valuation a satisfying $a-(b_2-2b_1)\equiv 2 \bmod 4$ is the valuation of a Galois translate ρ where the difference between $(\sigma\pm1)\alpha$ and ρ is an element $2\mu\in\mathfrak{P}_1^{i/4}$ where $(\sigma+1)\mu$ is in the basis for $\mathfrak{P}_0^{i/8}$. Each such a, therefore corresponds with the appearance of an $\mathcal{M}_1$. The count of such a is the same as the count for $A(4)$. The number of $\mathcal{M}$ that appear in $\mathfrak{P}_3^i$ is the same as the count for $A(3)$. Finally each valuation a satisfying $a\equiv 2 \bmod 4$ is the valuation of $(\sigma^2+1)\alpha=2\mu$ for some α in the basis for $\mathfrak{P}_2^{\lceil i/2\rceil}$. Also $(\sigma+1)\mu$ is in the basis for $\mathfrak{P}_0^{i/8}$, so each such a, therefore corresponds with the appearance of an $\mathcal{L}_3$. The count of such a is the same as the count for $A(6)$. Again, $A(5)$ counts the number of $\mathcal{L}$ in $\mathfrak{P}_3^i$.

Note the structure of $\mathfrak{P}_3^i$ under stable ramification is consistent with the structure of $\mathfrak{P}_3^i$ under unstable ramification so long as $b_2>4e_0-4b_1/3$.

Appendix A. The Modules

In this section we introduce twenty–three indecomposable $\mathbb{Z}_2[C_8]$-modules. It is left to the interested reader to translate our notation into Yakovlev's [Jak75].

Irreducibles: Four of the $\mathbb{Z}_2[C_8]$-modules are irreducible: $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ where $\mathcal{R}_n := \mathbb{Z}_2[\zeta_{2^n}]$, ζ_{2^n} denotes a primitive 2^n root of unity, and σ the generator of C_8 acts via multiplication by ζ_{2^n}.

The other nineteen modules are 'compounds'. They are organized according to fixed part – those fixed by σ^2 are listed first, followed by those fixed by σ^4, etc.

$\mathbb{Z}_2[C_2]$-*modules*: Besides the two irreducibles $\mathcal{R}_0$, $\mathcal{R}_1$, the group ring $\mathbb{Z}_2[\sigma]/\langle\sigma^2\rangle$ is the only other indecomposable module that is fixed by σ^2.

Notation for 'compounds': The group ring, $\mathbb{Z}_2[\sigma]/\langle\sigma^2\rangle$, is made up of two irreducibles. To make the relationships between irreducibles and their 'compounds' explicit, we will use diagrams like

$$\mathcal{R}_1 \to 1 \in \mathcal{R}_0$$

(instead of $\mathbb{Z}_2[\sigma]/\langle\sigma^2\rangle$). These diagrams are to be interpreted as follows: The number of $\mathbb{Z}_2[\sigma]$-generators is the number of irreducible modules that appear in the diagram. For example, $\mathcal{R}_1 \to 1 \in \mathcal{R}_0$ means two generators. Let us call them c and d. (*Think*: c generates $\mathcal{R}_1$ while d generates $\mathcal{R}_0$.) Relations determine the module. If there is no 'arrow' leaving an irreducible $\mathcal{R}_i$, then the trace $\Phi_{2^i}(\sigma)$ maps the generator to zero. So $\Phi_{2^0}(\sigma)d = 0$. Note $\Phi_{2^i}(x)$ denotes the cyclotomic polynomial and $x^8 - 1 = \Phi_{2^0}(x) \cdot \Phi_{2^1}(x) \cdot \Phi_{2^2}(x) \cdot \Phi_{2^3}(x)$. If there is an 'arrow' leaving an irreducible $\mathcal{R}_i$ (pointing to an element), then the trace $\Phi_{2^i}(\sigma)$ maps the generator to that element. In this case $\Phi_{2^1}(\sigma)c = 1 \cdot d$.

$\mathbb{Z}_2[C_4]$-*modules*: There are three indecomposable modules fixed by σ^4 (yet not fixed by σ^2). Notation for two other decomposable modules is included as it will be needed to describe certain modules later (those not fixed by σ^4). For three (of these five), the submodule fixed by σ^2 is the group ring $\mathbb{Z}_2[\sigma]/\langle\sigma^2\rangle$ (note how their diagams include $\mathcal{R}_1 \to 1 \in \mathcal{R}_0$):

$$(\mathcal{G})\colon \mathcal{R}_2 \to 1 \in \mathcal{R}_1 \to 1 \in \mathcal{R}_0, \quad (\mathcal{H})\colon \begin{array}{ccc} \mathcal{R}_2 & & \\ & \searrow & \\ \mathcal{R}_1 & \to & 1 \in \mathcal{R}_0 \end{array}, \quad (\mathcal{I})\colon \mathcal{R}_2 \oplus (\mathcal{R}_1 \to 1 \in \mathcal{R}_0).$$

Denote the three generators by b, c, d. (*Think*: generating $\mathcal{R}_2, \mathcal{R}_1, \mathcal{R}_0$, respectively.) Recall $(\sigma - 1)d = 0$ while $(\sigma + 1)c = d$. In $\mathcal{G}$, we have $\Phi_{2^2}(\sigma)b = 1 \cdot c$. So $\mathcal{G}$ is the group ring $\mathbb{Z}_2[\sigma]/\langle\sigma^4\rangle$. In $\mathcal{H}$, we have $\Phi_{2^2}(\sigma)b = 1 \cdot d$. While in $\mathcal{I}$, $\Phi_{2^2}(\sigma)b = 0$.

For two (of these five), the submodule fixed by σ^2 is the maximal order of $\mathbb{Z}_2[\sigma]/\langle\sigma^2\rangle$ (note how $\mathcal{R}_1 \oplus \mathcal{R}_0$ appears):

$$(\mathcal{L}):\ \mathcal{R}_2 \ \to \ \begin{cases} 1 \in \mathcal{R}_1 \\ \oplus \\ 1 \in \mathcal{R}_0 \end{cases}, \quad (\mathcal{M}) : \mathcal{R}_2 \oplus \mathcal{R}_1 \oplus \mathcal{R}_0.$$

Denote the three generators by b, c, d where $(\sigma - 1)d = 0$ and $(\sigma + 1)c = 0$. In $\mathcal{L}$, we have $\Phi_{2^2}(\sigma)b = 1 \cdot c + 1 \cdot d$. In $\mathcal{M}$, we have $\Phi_{2^2}(\sigma)b = 0$. So $\mathcal{M}$ is the maximal order of $\mathbb{Z}_2[\sigma]/\langle \sigma^4 \rangle$.

$\mathbb{Z}_2[C_8]$-*modules*: The remaining fifteen indecomposable modules can now be listed. They are collected according to submodule fixed by σ^4.
Fixed part $\mathcal{G}$.

$$(\mathcal{G}_1) : \mathcal{R}_3 \to 1 \in \mathcal{R}_2 \to 1 \in \mathcal{R}_1 \to 1 \in \mathcal{R}_0 \quad (\mathcal{G}_3) : \begin{array}{ccccc} \mathcal{R}_3 & & & & \\ & \searrow & & & \\ \mathcal{R}_2 & \to & 1 \in \mathcal{R}_1 & \to & 1 \in \mathcal{R}_0 \end{array}$$

$$(\mathcal{G}_2) : \mathcal{R}_3 \to \lambda \in \mathcal{R}_2 \to 1 \in \mathcal{R}_1 \to 1 \in \mathcal{R}_0 \quad (\mathcal{G}_4) : \begin{array}{ccccc} & & \mathcal{R}_3 & & \\ & & & \searrow & \\ \mathcal{R}_2 & \to & 1 \in \mathcal{R}_1 & \to & 1 \in \mathcal{R}_0 \end{array}$$

Call the generators a, b, c, d, where the $\mathbb{Z}_2[\sigma]$-relations among b, c, d are as in $\mathcal{G}$. In $\mathcal{G}_1$, we have $\Phi_{2^3}(\sigma)a = 1 \cdot b$. So $\mathcal{G}_1$ is the group ring $\mathbb{Z}_2[\sigma]$. In $\mathcal{G}_2$, we have $\Phi_{2^3}(\sigma)a = \lambda \cdot b$ where $\lambda = \sigma - 1$. In $\mathcal{G}_3$, $\Phi_{2^3}(\sigma)a = 1 \cdot c$. In $\mathcal{G}_4$, $\Phi_{2^3}(\sigma)a = 1 \cdot d$.
Fixed part $\mathcal{H}$.

$$(\mathcal{H}_1) : \ \mathcal{R}_3 \ \to \left\{ \begin{array}{lll} \lambda \in \mathcal{R}_2 & & \\ \oplus & \searrow & \\ 1 \in \mathcal{R}_1 & \to & 1 \in \mathcal{R}_0 \end{array} \right. \qquad (\mathcal{H}_2) : \ \begin{array}{ccc} \mathcal{R}_3 & & \\ & \searrow & \\ \mathcal{R}_2 & \to & 1 \in \mathcal{R}_0 \\ & \nearrow & \\ \mathcal{R}_1 & & \end{array}$$

Call the generators a, b, c, d, where the $\mathbb{Z}_2[\sigma]$-relationships among b, c, d are as in $\mathcal{H}$. In $\mathcal{H}_1$, $\Phi_{2^3}(\sigma)a = \lambda \cdot 1 \cdot b + 1 \cdot c$. In $\mathcal{H}_2$, $\Phi_{2^3}(\sigma)a = d$.
Fixed part $\mathcal{I}$.

$$(\mathcal{I}_1) : \ \mathcal{R}_3 \ \to \left\{ \begin{array}{lll} 1 \in \mathcal{R}_2 & & \\ \oplus & & \\ 1 \in \mathcal{R}_1 & \to & 1 \in \mathcal{R}_0 \end{array} \right. \qquad (\mathcal{I}_2) : \ \begin{array}{cc} \mathcal{R}_3 & \to \\ \mathcal{R}_1 & \to \end{array} \left\{ \begin{array}{l} 1 \in \mathcal{R}_2 \\ \oplus \\ 1 \in \mathcal{R}_0 \end{array} \right.$$

Each module is generated by a, b, c, d, where the $\mathbb{Z}_2[\sigma]$-relationships among b, c, d are as in $\mathcal{I}$. In $\mathcal{I}_1$, $\Phi_{2^3}(\sigma)a = 1 \cdot b + 1 \cdot c$. In $\mathcal{I}_2$, $\Phi_{2^3}(\sigma)a = 1 \cdot b + 1 \cdot d$.
Fixed part $\mathcal{L}$ or $\mathcal{M}$.

$$(\mathcal{L}_1) : \ \mathcal{R}_3 \ \to \ 1 \in \mathcal{R}_2 \ \to \left\{ \begin{array}{l} 1 \in \mathcal{R}_1 \\ \oplus \\ 1 \in \mathcal{R}_0 \end{array} \right. \qquad (\mathcal{L}_3) : \ \begin{array}{ccc} \mathcal{R}_3 & & \\ & \searrow & \\ \mathcal{R}_2 & \to & \left\{ \begin{array}{l} 1 \in \mathcal{R}_1 \\ \oplus \\ 1 \in \mathcal{R}_0 \end{array} \right. \end{array}$$

$$(\mathcal{L}_2) : \ \mathcal{R}_3 \ \to \ \lambda \in \mathcal{R}_2 \ \to \left\{ \begin{array}{l} 1 \in \mathcal{R}_1 \\ \oplus \\ 1 \in \mathcal{R}_0 \end{array} \right. \qquad (\mathcal{M}_1) : \ \mathcal{R}_3 \ \to \left\{ \begin{array}{l} \lambda \in \mathcal{R}_2 \\ \oplus \\ 1 \in \mathcal{R}_1 \\ \oplus \\ 1 \in \mathcal{R}_0 \end{array} \right.$$

The generators are a, b, c, d, where the $\mathbb{Z}_2[\sigma]$-relationships among b, c, d are as in $\mathcal{L}$ or $\mathcal{M}$ respectively. In $\mathcal{L}_1$, $\Phi_{2^3}(\sigma)a = b$. In $\mathcal{L}_2$, $\Phi_{2^3}(\sigma)a = \lambda \cdot b$. In $\mathcal{L}_3$, $\Phi_{2^3}(\sigma)a = 1 \cdot c + 1 \cdot d$. In $\mathcal{M}_1$, $\Phi_{2^3}(\sigma)a = 1 \cdot b + 1 \cdot c + 1 \cdot d$.
Hybrids of $\mathcal{H}_1$. The next three modules result from the linking of an $\mathcal{H}_1$ with either another $\mathcal{R}_3$, or with a $\mathcal{G}$, or with a $\mathcal{L}$.

$$
(\mathcal{H}_{1,2}): \quad \begin{array}{ccccc} & & \mathcal{R}_3 & & \\ & & & \searrow & \\ \mathcal{R}_3 & \rightarrow & \left\{ \begin{array}{l} 1 \in \mathcal{R}_1 \\ \oplus \\ \lambda \in \mathcal{R}_2 \end{array} \right. & \begin{array}{c} \rightarrow \\ \nearrow \\ \ \end{array} & 1 \in \mathcal{R}_0 \end{array}
$$

This module is generated by a_1, a_2, b, c, d with the $\mathbb{Z}_2[\sigma]$-relationships among b, c, d as in $\mathcal{H}$, while $\Phi_{2^3}(\sigma)a_1 = \lambda \cdot b + 1 \cdot c$ and $\Phi_{2^3}(\sigma)a_2 = d$. If $\Phi_{2^3}(\sigma)a_1 = 0$, $\mathcal{H}_2$ would decompose off. If $\Phi_{2^3}(\sigma)a_2 = 0$, $\mathcal{H}_1$ would decompose off. It is a mixture of $\mathcal{H}_1$ and $\mathcal{H}_2$, hence the name.

$$
(\mathcal{H}_1\mathcal{G}): \quad \begin{array}{ccccc} \mathcal{R}_3 & \rightarrow & \left\{ \begin{array}{l} 1 \in \mathcal{R}_1 \\ \oplus \\ \lambda \in \mathcal{R}_2 \end{array} \right. & \begin{array}{c} \rightarrow \\ \\ \rightarrow \end{array} & \begin{array}{c} 1 \in \mathcal{R}_0 \\ \\ \oplus \end{array} \\ & & & & \\ \mathcal{R}_2 & \rightarrow & 1 \in \mathcal{R}_1 & \rightarrow & 1 \in \mathcal{R}_0 \end{array}
$$

This module is generated by a_1, b_1, c_1, d_1 and b_2, c_2, d_2. The $\mathbb{Z}_2[\sigma]$-relationships among b_2, c_2, d_2 are as in $\mathcal{G}$. The $\mathbb{Z}_2[\sigma]$-relationships among a_1, c_1, d_1 are as in $\mathcal{H}_1$ with $(\sigma^2 + 1)b_1 = 1 \cdot d_1 + 1 \cdot d_2$.

$$
(\mathcal{H}_1\mathcal{L}): \quad \begin{array}{ccccc} \mathcal{R}_3 & \rightarrow & \left\{ \begin{array}{l} 1 \in \mathcal{R}_1 \\ \oplus \\ \lambda \in \mathcal{R}_2 \end{array} \right. & \begin{array}{c} \rightarrow \\ \\ \rightarrow \end{array} & \begin{array}{c} 1 \in \mathcal{R}_0 \\ \\ \oplus \end{array} \\ & & & & \\ & & \mathcal{R}_2 & \rightarrow & \left\{ \begin{array}{l} 1 \in \mathcal{R}_0 \\ \oplus \\ 1 \in \mathcal{R}_1 \end{array} \right. \end{array}
$$

This module is generated by a_1, b_1, c_1, d_1 and b_2, c_2, d_2. The $\mathbb{Z}_2[\sigma]$-relationships among b_2, c_2, d_2 are as in $\mathcal{L}$. The $\mathbb{Z}_2[\sigma]$-relationships among a_1, c_1, d_1 are as in $\mathcal{H}_1$ with $(\sigma^2 + 1)b_1 = 1 \cdot d_1 + (1 \cdot c_2 + 1 \cdot d_2)$.

Appendix B. The Bases by Case, A through H

From §3.4, we inherit sequences of elements ordered in terms of increasing valuation (for Case A, we have $\ldots \rho, \overline{2\rho}, (\sigma^2+1)\alpha, \overline{2(\sigma^2+1)\alpha}, 2\alpha, \overline{4\alpha}, (\sigma+1)(\sigma^2+1)\alpha, \overline{2(\sigma+1)(\sigma^2+1)\alpha}, 2\rho, \ldots$). Following §2.2.3, we are interested in those elements 'in view' (*i.e.* with valuation in $i, i+1, \ldots, i+v_3(2)-1$). As we vary m the 'view' changes. Indeed, for each case, there are eight views (eight sets). They are listed below. Recall from §2.2.3 it is easy to determine the subscripts m associated with a particular 'view'. For example, the elements in $A(2)$ appear for $i \leq v_3\left(\overline{(\sigma+1)(\sigma^2+1)\alpha}\right)$ and $v_3\left((\sigma+1)(\sigma^2+1)\alpha\right) \leq 8e_0 + i - 1$. In other words, $\lceil (i + b_3 - 4b_1 - 4b_2)/8 \rceil \leq m \leq \lceil (i + 8e_0 - 4b_1 - 4b_2)/8 \rceil - 1$.

Case A

(1) $\rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha,\ \overline{4\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma+1)(\sigma^2+1)\alpha}$

(2) $\overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha,\ \overline{4\alpha},\ (\sigma+1)(\sigma^2+1)\alpha$

(3) $\frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha,\ \overline{4\alpha}$

(4) $\overline{2\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha$

(5) $\alpha,\ \overline{2\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha}$

(6) $\overline{(\sigma^2+1)\alpha},\ \alpha,\ \overline{2\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha$

(7) $\frac{1}{2}(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \alpha,\ \overline{2\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho}$

(8) $\overline{\rho},\ \frac{1}{2}(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \alpha,\ \overline{2\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho$

Case B

(1) $\rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{4\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha}$

(2) $\overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{4\alpha}$

(3) $\overline{2\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha,\ (\sigma+1)(\sigma^2+1)\alpha$

(4) $\frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ 2\alpha$

(5) $\alpha,\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha}$

(6) $\overline{(\sigma^2+1)\alpha},\ \alpha,\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha$

(7) $\frac{1}{2}(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \alpha,\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho,\ \overline{2\rho}$

(8) $\overline{\rho},\ \frac{1}{2}(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \alpha,\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \rho$

Case C

(1) $\rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ 2\alpha,\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ \overline{4\alpha}$

(2) $\overline{2\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ 2\alpha,\ \overline{2(\sigma+1)(\sigma^2+1)\alpha}$

(3) $\overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ 2\alpha$

(4) $\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha$

(5) $\frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha}$

(6) $\overline{(\sigma^2+1)\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ \overline{2\rho},\ (\sigma^2+1)\alpha$

(7) $\frac{1}{2}(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ \overline{2\rho}$

(8) $\overline{\rho},\ \frac{1}{2}(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho$

Case D

(1) $\rho,\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ 2\alpha,\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ \overline{4\alpha}$

(2) $\overline{2\alpha},\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ 2\alpha,\ \overline{2(\sigma+1)(\sigma^2+1)\alpha}$

(3) $\overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ 2\alpha$

(4) $\alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ (\sigma+1)(\sigma^2+1)\alpha$

(5) $\frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha}$

(6) $\overline{(\sigma^2+1)\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\rho}$

(7) $\overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho,\ (\sigma^2+1)\alpha$

(8) $\frac{1}{2}(\sigma^2+1)\alpha,\ \overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \alpha,\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \overline{2\alpha},\ \rho$

Case E

(1) $\overline{2\alpha},\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha,\ 2\rho$

(2) $\rho,\ \overline{2\alpha},\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha$

(3) $\alpha,\ \rho,\ \overline{2\alpha},\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha}$

(4) $\overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha}$

(5) $\overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ \overline{2\rho},\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha$

(6) $\frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ \overline{2\rho},\ (\sigma^2+1)\alpha$

(7) $\frac{1}{2}(\sigma^2+1)\alpha,\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ \overline{2\rho}$

(8) $\overline{\rho},\ \frac{1}{2}(\sigma^2+1)\alpha,\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha}$

Case F

(1) $\overline{2\alpha},\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha,\ 2\rho$

(2) $\rho,\ \overline{2\alpha},\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha$

(3) $\alpha,\ \rho,\ \overline{2\alpha},\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha}$

(4) $\overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2(\sigma^2+1)\alpha}$

(5) $\overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ (\sigma^2+1)\alpha,\ \overline{2\rho},\ (\sigma+1)(\sigma^2+1)\alpha$

(6) $\frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ (\sigma^2+1)\alpha,\ \overline{2\rho}$

(7) $\overline{\rho},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha},\ (\sigma^2+1)\alpha$

(8) $\frac{1}{2}(\sigma^2+1)\alpha,\ \overline{\rho},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ \overline{2\alpha}$

Case G

$$
\begin{array}{ll}
(1) & (\sigma^2+1)\alpha,\ \overline{2\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha,\ 2\rho \\
(2) & \rho,\ (\sigma^2+1)\alpha,\ \overline{2\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha \\
(3) & \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha} \\
(4) & \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha} \\
(5) & \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\alpha},\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\rho} \\
(6) & \overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\alpha},\ (\sigma+1)(\sigma^2+1)\alpha \\
(7) & \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ \overline{2\alpha} \\
(8) & \overline{\alpha},\ \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha
\end{array}
$$

Case H

$$
\begin{array}{ll}
(1) & (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha,\ 2\rho \\
(2) & \rho,\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha},\ 2\alpha \\
(3) & \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha},\ \overline{2(\sigma+1)(\sigma^2+1)\alpha} \\
(4) & \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{2\rho},\ \overline{2(\sigma^2+1)\alpha} \\
(5) & \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha},\ \overline{2\rho} \\
(6) & \overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha,\ \overline{2\alpha} \\
(7) & \overline{\alpha},\ \overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha,\ (\sigma+1)(\sigma^2+1)\alpha \\
(8) & \frac{1}{2}(\sigma+1)(\sigma^2+1)\alpha,\ \overline{\alpha},\ \overline{\rho},\ \overline{(\sigma^2+1)\alpha},\ \overline{(\sigma+1)(\sigma^2+1)\alpha},\ \alpha,\ \rho,\ (\sigma^2+1)\alpha
\end{array}
$$

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